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The classifying space of a monoid

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0 Introduction

Classifying spaces come up in at least two contexts. Firstly, it is a construction which assigns to a group a simplicial complex which reflects its structure in the following way. The simplicial complex has a single 0-simplex and each element of the group is represented by a 1-simplex that starts and ends in the unique 0-simplex. Furthermore, there is an n-simplex for every sequence of n elements of the group. The faces of an n-simplex are the shorter sequences obtained by multiplying subsequent elements. A sequence that contains trivial elements is identified with the shorter sequence obtained by removing the trivial elements.

Secondly, a classifying space can also be defined for a small category. It is the simplicial complex which contains a 0-simplex for every object of the category, a 1-simplex for every morphism, and in general an n-simplex for every sequence of n composable morphisms. The faces of an n-simplex are the shorter sequences obtained by composing subsequent morphisms. A sequence that contains identity morphisms is identified with the shorter sequence obtained by removing these.

These two definitions don't just look similar, the former is a special case of the latter, if we view a group as a category with one object and a morphism for every element, and define composition as multiplication.

It has been shown that every group is the fundamental group of its classifying space, and that the other homotopy groups of the classifying space are trivial. This result is more significant than it might first seem, for this determines the space up to homotopy.

This naturally brings up the question whether we can say more about the homotopy type of the classifying space of a category. The next obvious class of cases to consider is formed by monoids. These are algebraic structures similar to groups, but without the requirement that elements have inverses. Like groups, monoids can be considered as categories with one object — in fact, any category with one object is a monoid viewed in this way.

There is a standard way of turning a monoid into a group: its groupification. The fundamental group of the classifying space of a monoid is its groupification. This fact can be found in the literature, e.g. in [WEIBEL to appear], as Application 3.4.3 of Lemma 3.4. In this thesis I will give a direct proof of the fact that the groupification map induces an isomorphism between the fundamental groups of the classifying spaces of a monoid and its groupification. Furthermore, for commutative monoids and free monoids I will prove that the groupification map actually induces a homotopy equivalence between classifying spaces. This is inspired by [RABRENOVIĆ 2005], where the result is proved for the monoid of natural numbers $(\mathbb{N}, +, 0)$ and for monoids $(M, \cdot, 1)$ with a distinguished element z such that for every element m of M, both zm and mz equal z.

The classifying space construction is functorial. In Section 1, this functor will be defined in two steps. First we will define the nerve functor which assigns to a category a simplicial set, its nerve. Then we will define the topological realisation functor, which turns a simplicial set into a topological space.

In Section 2, we will give a number of properties of the classifying space functor, which we will need to prove the main theorems of this thesis. It will be shown that through the classifying space functor, natural transformations between functors induce homotopies between continuous maps. The section ends with a number of important theorems by Daniel Quillen.

Section 3 will exhibit the little amount of monoid theory necessary for the subsequent results. It contains some examples of monoids, and a definition of cancellativisation and groupification.

In Section 4, we will first restate the characterisation of the classifying space of a group known from the literature. We will then prove that the fundamental group of a monoid is its groupification and that for commutative monoids and free monoids, the groupification map induces a homotopy equivalence between classifying spaces.

Finally, in Section 5, we will briefly consider how one might proceed onwards.

1 The classifying space construction

The *classifying space* functor assigns to each small category a topological space, its *classifying space*. We will construct the functor in two stages, through the aid of *simplices*, which have both a categorical and a topological interpretation. In the first stage, we disassemble a category into simplices, while remembering the combinatorial relations between them. The structure which we use to retain this information is called a *simplicial set*, and we will denote the relevant category of *simplicial sets* by $\mathfrak{Set}^{\nabla}$. The functor we thus get from $\mathfrak{Cat}$ (the category of small categories and functors) to $\mathfrak{Set}^{\nabla}$ is called the *nerve* functor N.

In the second stage, using the combinatorial instructions, we re-assemble the simplices into a topological space. This space is always a *Kelley space* (compactly generated and Hausdorff) and we get the *topological realisation* functor $|-|$ from $\mathfrak{Set}^{\nabla}$ to $\mathfrak{Kel}$, the category of Kelley spaces and continuous maps. We don't take $\mathfrak{Top}$ as the target category, because then the topological realisation functor would not preserve finite limits (see Subsection 2.2).

$$\mathfrak{Cat} \xrightarrow{\quad N \quad} \mathfrak{Set}^{\nabla} \xrightarrow{\quad |-| \quad} \mathfrak{Kel}$$

Readers left dissatisfied by any aspect of what follows may consult sections I.1 and I.2 of [GOERSS & JARDINE 1999], where the classifying space functor is constructed in a somewhat more concise fashion and in a wider context, but along similar lines.

1.1 Simplices

The key to the classifying space construction is the simplex category. As simplices have several different interpretations, the simplex category can be defined in different yet equivalent ways. Perhaps the most straightforward way is to define simplices as finite ordered sets.

Definition 1.1.1 The *simplex category* Δ is the category whose objects are the sets $\{0, 1, \ldots, n\}$, with the canonical order $\leq$, for all $n \geq -1$, and whose morphisms are the order-preserving maps between them. We write ∇ for Δ^{op}.

We will denote an object $\{0, 1, \ldots, n\}$ of Δ by the natural number $n+1$. In particular, 0 is the empty set and 1 a singleton. It will be useful to distinguish several types of morphisms in Δ. A map $f\colon n \longrightarrow m$ in Δ is called a face map if $n < m$ and a degeneracy map if $n > m$. As a category, Δ is generated by all injective face maps $\delta_i\colon n \longrightarrow n+1$ (called *generating face maps*) and all surjective degeneracy maps $\sigma_i\colon n+1 \longrightarrow n$ (called *generating degeneracy maps*).

A set with a transitive and reflexive relation $\leq$ can be viewed as a category whose objects are its elements and where for any objects x and y there is a unique morphism from x to y if and only if $x \leq y$. Furthermore, relation preserving maps between such sets correspond precisely to functors between the respective categories. This gives us a functor $\Delta \longrightarrow \mathfrak{Cat}$ which is injective on both objects and morphisms. For any finite ordinal n, denote its image in $\mathfrak{Cat}$ in fraktur: $\mathfrak{n}$. Specifically, $\mathfrak{0}$ is the empty category, $\mathfrak{1}$ is a category with one object and only the identity morphism and $\mathfrak{2}$ is a category with two objects and only one non-trivial morphism, which connects the two objects.

Next, we construct a functor $\Delta \longrightarrow \mathfrak{Rel}$ which is injective on both objects and morphisms. Any ordinal n we send to the *standard* $n-1$*-simplex* Δ_{n-1}, that is the convex subset of $\mathbb{R}^n$ spanned by its basis vectors $e_1, e_2, \ldots, e_n$ (the *vertices* of Δ_{n-1}). Any order-preserving map between ordinals n and m we send to the linear map from Δ_n to Δ_m induced by the corresponding order preserving map between the ordered vertex-sets $(e_1, e_2, \ldots, e_n)$ and $(e_1, e_2, \ldots, e_m)$.

From now on, we will at times identify Δ with its image in $\mathfrak{Cat}$ and at times with its image in $\mathfrak{Rel}$.

> **Definition 1.1.2** A *simplicial set* is a contravariant functor from Δ to $\mathfrak{Set}$, that is, a functor from ∇ to $\mathfrak{Set}$. Denote the category of all simplicial sets and natural transformations between them by $\mathfrak{Set}^\nabla$.

For any simplicial set X, and for any $n \in \mathbb{N}$, we write X_n for $X(n)$, and we call any element $x \in X_n$ an n-simplex of X.

1.2 The nerve of a category

The definition of the nerve of a category is very succinct:

> **Definition 1.2.1** The *nerve functor* $N\colon \mathfrak{Cat} \longrightarrow \mathfrak{Set}^\nabla$ is the functor $\mathcal{C} \longmapsto \mathrm{Funct}(-, \mathcal{C})$.

> **Remark 1.2.2** The Nerve functor is the *dual Yoneda functor* of the dual Yoneda lemma, composed with the restriction $\mathfrak{Set}^{\mathfrak{Cat}} \longrightarrow \mathfrak{Set}^\nabla$.

An n-simplex of the nerve of a category $\mathcal{C}$ is a sequence of $n-1$ composable morphisms of $\mathcal{C}$, connecting n objects. For any $1 \leq k \leq n-2$, the k^{th} generating face map sends an n-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} X_k \xrightarrow{f_k} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

to the $n-1$-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_k f_{k-1}} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

obtained by composing two subsequent morphisms and omitting the intermediate object. The 0^{th} generating face map sends an n-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

to the $n-1$ simplex

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

and the $n-1^{\text{th}}$ generating face map sends an n-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-3}} X_{n-2} \xrightarrow{f_{n-2}} X_{n-1}$$

to the $n-1$ simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-3}} X_{n-2}.$$

For any $0 \leq k \leq n-1$, the k^{th} generating degeneracy map sends an n-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} X_k \xrightarrow{f_k} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

to the $n+1$-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} X_k \xrightarrow{\text{id}} X_k \xrightarrow{f_k} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

obtained by inserting the identity morphism of the object X_k. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is sent to the morphism which sends an n-simplex

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} X_{n-1}$$

of the nerve of $\mathcal{C}$ to the n-simplex

$$F(X_0) \xrightarrow{F(f_0)} F(X_1) \xrightarrow{F(f_1)} \cdots \xrightarrow{F(f_{n-2})} F(X_{n-1})$$

of the nerve of $\mathcal{D}$.

1.3 The topological realisation of a simplicial set

The definition of the topological realisation functor takes a bit more work than the definition of the nerve functor — it will be constructed as the composition of two functors. The reason for this is that we will define the topological realisation of a simplicial set (the gluing back together of simplices) as a colimit in $\mathfrak{Kel}$, so we require an auxiliary functor to turn the simplicial set into a functor from an index category in which simplices are objects in their own right to Δ. Since Δ can be viewed as a subcategory of $\mathfrak{Kel}$, we can subsequently take colimits. The appropriate intermediate category is the slice category over Δ in $\mathfrak{Cat}$, denoted by $\text{id}\,/\Delta$. The objects of $\text{id}\,/\Delta$ are functors from any other small category $\mathcal{C}$ to Δ, and a morphism between any such functors $F\colon \mathcal{C} \longrightarrow \Delta$ and $G\colon \mathcal{D} \longrightarrow \Delta$ is a functor $H\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $F = G \circ H$. We will thus define the topological realisation functor as the composition of a functor T from $\mathfrak{Set}^{\nabla}$ to $\text{id}\,/\Delta$ and a functor L from $\text{id}\,/\Delta$ to $\mathfrak{Kel}$.

$$\mathfrak{Set}^{\nabla} \xrightarrow{\quad T \quad} \text{id}\,/\Delta \xrightarrow{\quad L \quad} \mathfrak{Kel}$$

Definition 1.3.1 Let $T\colon \mathfrak{Set}^{\nabla} \longrightarrow \mathrm{id}/\Delta$ be the following functor. For every simplicial set X, let the source of $T(X)$ be the category $\mathcal{I}_X$ whose objects are given by $\coprod_{n\in\mathbb{N}} X_{\mathfrak{n}}$, and $\mathrm{mor}(x,y) = \{f \in \mathrm{Mor}(\nabla) | X_f(y) = x\}$. $T(X)$ then sends $x \in X_{\mathfrak{n}}$ to $\mathfrak{n}$ and f to f.

For $\{\mu_{\mathfrak{i}}\}$ a natural transformation $X \longrightarrow Y$ in $\mathfrak{Set}^{\nabla}$, $T(\{\mu_{\mathfrak{i}}\})\colon T(X) \longrightarrow T(Y)$ is the functor $\mathcal{I}(X) \longrightarrow \mathcal{I}(Y)$ which sends $x \in X_{\mathfrak{n}}$ to $\mu_{\mathfrak{n}}(x)$ and a morphism f to itself.

This definition is well-defined since by the definition of a natural transformation, for every $n \in \mathbb{N}$, the following diagram commutes:

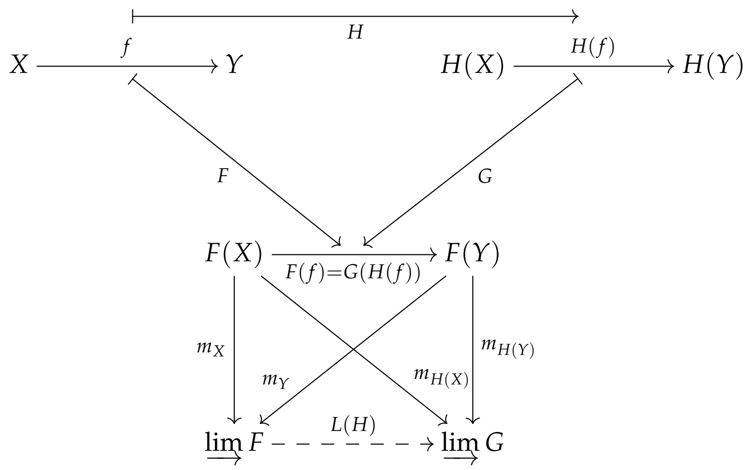
$$\begin{tikzcd} X_{\mathfrak{n}} \arrow[r, "\mu_{\mathfrak{n}}"] \arrow[d, "X_f"] & Y_{\mathfrak{n}} \arrow[d, "Y_f"] \\ X_{\mathfrak{m}} \arrow[r, "\mu_{\mathfrak{m}}"] & Y_{\mathfrak{m}} \end{tikzcd}$$

Also, $T(\{\mu_{\mathfrak{i}}\})$ is a morphism in id/Δ (as claimed) because the following diagram commutes:

$$\begin{tikzcd} \mathcal{I}(X) \arrow[rr, "T(\{\mu_{\mathfrak{i}}\})"] \arrow[dr, "T(X)"'] & & \mathcal{I}(Y) \arrow[dl, "T(Y)"] \\ & \Delta & \end{tikzcd}$$

As promised, we get from id/Δ to $\mathfrak{Rel}$ by embedding Δ into $\mathfrak{Rel}$ and taking the colimit. However, since we are coming from the slice category over Δ and not a specific category of functors with a fixed source, we have to make sure that taking colimits is functorial.

Any object G of id/Δ is a functor $\mathcal{D} \longrightarrow \Delta$, with $\mathcal{D}$ a small category. Viewing Δ as a subcategory of $\mathfrak{Rel}$, we can take the colimit of G, and the definition of the colimit then gives us maps $m_Y\colon G(Y) \longrightarrow \varinjlim G$ that commute with maps $G(f)$, for all objects Y and morphisms f of $\mathcal{D}$. Now, for any other object $F\colon \mathcal{C} \longrightarrow \Delta$ of id/Δ and any morphism $H\colon \mathcal{C} \longrightarrow \mathcal{D}$ from F to G, by definition it is true that $F(f) = G(H(f))$ for every morphism f of $\mathcal{C}$. Hence for every object X of $\mathcal{C}$, the map $m_{H(X)}$ is also a map from $F(X)$ to $\varinjlim G$ and these maps commute with maps $F(f)$. The universal property of the colimit then gives us a unique map $L(H)\colon \varinjlim F \longrightarrow \varinjlim G$, which we will appeal to in the definition of our functor L from id/Δ to $\mathfrak{Rel}$.



> **Definition 1.3.2** $L\colon \mathrm{id}/\Delta \longrightarrow \mathfrak{Kel}$ is the functor which sends an object $F\colon \mathcal{C} \longrightarrow \Delta$ of id/Δ to $\varinjlim F$ in $\mathfrak{Kel}$, and a morphism $H\colon \mathcal{C} \longrightarrow \mathcal{D}$ from F to another object G to the canonical map $\varinjlim F \longrightarrow \varinjlim G$.

> **Definition 1.3.3** The *topological realisation* functor is the composed functor $|-| = L \circ T\colon \mathfrak{Set}^{\nabla} \longrightarrow \mathfrak{Kel}$.

In the topological realisation of a simplicial set, we have transformed abstract simplices into topological simplices. Taking the colimit translates face maps into inclusion relations between these topological simplices, and through degeneracy maps degenerate simplices are identified with lower-dimensional analogues.

Since for every n, the standard n-simplex Δ_n is homeomorphic to the n-ball, the topological realisation of a simplicial set is a CW-complex (with the non-degenerate simplices as its cells). For a precise proof of this fact, see Theorem 1 of [MILNOR 1957] (which uses the old terminology of *semi-simplicial complex* for *simplicial set*).

> **Definition 1.3.4** The *classifying space* functor is the composed functor $|-| \circ N\colon \mathfrak{Cat} \longrightarrow \mathfrak{Kel}$.

The classifying space functor is usually denoted B. However, for the sake of keeping proofs in the following section readable, I have decided to denote it by $|-|$ as well, which should not lead to confusion.

2 Properties of the classifying space functor

2.1 Adjointness and the resulting preservation of limits

In this subsection, we will prove the following theorem:

Theorem 2.1.1 *The nerve functor commutes with limits and the topological realisation functor commutes with colimits.*

We will show that the nerve functor has a left adjoint and the topological realisation functor a right adjoint — Theorem 2.1.1 then automatically follows.

Before we can proceed, we must first define the *comma* and *cocomma* categories of a functor. Comma and cocomma categories are the subject of Quillen's theorems A and B presented in Subsection 2.3, and as such will play an important part in Subsection 4.3. However, we already need them in the proof of the next theorem, in order to establish that the topological realisation functor has a right adjoint. Comma and cocomma categories can be defined in various degrees of generality, the following will be sufficiently general for our purposes.

Definition 2.1.2 Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ a functor. For every object Y of $\mathcal{D}$, the *comma category F/Y of F over Y* is defined as follows. Its objects are pairs (X, g), with X an object of $\mathcal{C}$ and $g\colon F(X) \longrightarrow Y$ a morphism of $\mathcal{D}$. A morphism between any such objects (X_1, g_1) and (X_2, g_2) is a morphism f in $\mathcal{C}$ such that $g_1 = g_2 F(f)$.

$$\begin{tikzcd} F(X_1) \arrow[rr, "F(f)"] \arrow[dr, "g_1"'] & & F(X_2) \arrow[dl, "g_2"] \\ & Y & \end{tikzcd}$$

Definition 2.1.3 Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ a functor. For every object Y of $\mathcal{D}$, the *cocomma category $Y\backslash F$ of F over Y* is defined as follows. Its objects are pairs (X, g), with X an object of $\mathcal{C}$ and $g\colon Y \longrightarrow F(X)$ a morphism of $\mathcal{D}$. A morphism between any such objects (X_1, g_1) and (X_2, g_2) is a morphism f in $\mathcal{C}$ such that $F(f)g_1 = g_2$.

$$\begin{tikzcd} & Y \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ F(X_1) \arrow[rr, "F(f)"] & & F(X_2) \end{tikzcd}$$

The comma category of an identity functor is a special case, this is the *slice category* which we encountered in Subsection 1.3 (similarly the cocomma category of an identity functor is called the *coslice category*).

Our definition of the nerve functor only required the inclusion functor $\Delta \longrightarrow \mathfrak{Cat}$ and the fact that $\mathrm{Funct}(\mathfrak{n}, \mathcal{C})$ is a set for every object $\mathfrak{n}$ of Δ and every object $\mathcal{C}$ of $\mathfrak{Cat}$. It is therefore possible to define an analogous functor for any injective functor I from any category $\mathcal{D}_1$ to any category $\mathcal{C}$ such that $\mathrm{Mor}_{\mathcal{C}}(I(X), Y)$ is a set for every object X of $\mathcal{D}_1$ and every object Y of $\mathcal{C}$.

Similarly, our definition of the topological realisation functor only used the inclusion functor $\Delta \longrightarrow \mathfrak{Kel}$ and the fact that $\mathfrak{Kel}$ has colimits. It is therefore possible to define an analogous functor for any injective functor J between any category $\mathcal{D}_2$ and any category $\mathcal{K}$ which has colimits.

A functor of the first type is composable with a functor of the latter type if and only if $\mathcal{D}_1 = \mathcal{D}_2$.

Specifically, we are interested in the reversal of the situation we started out with, taking $\mathcal{C} = \mathfrak{Kel}$, $\mathcal{D}_1 = \mathcal{D}_2 = \Delta$ and $\mathcal{K} = \mathfrak{Cat}$, with $I = n \longmapsto \Delta_n$ and $J = n \longmapsto \mathfrak{n}$. Adapting Definition 1.2.1 appropriately, we get the *singular functor* $S\colon \mathfrak{Kel} \longrightarrow \mathfrak{Set}^{\nabla}$.

Definition 2.1.4 The *singular functor* $S\colon \mathfrak{Kel} \longrightarrow \mathfrak{Set}^{\nabla}$ is the functor $X \longmapsto \mathrm{Cont}(-, X)$.

Recall that the topological realisation functor was defined as the composition of the functor T defined in Definition 1.3.1 and the colimit functor L in Definition 1.3.2. We get the *categorical realisation functor* $C\colon \mathfrak{Set}^{\nabla} \longrightarrow \mathfrak{Cat}$ by taking colimits in $\mathfrak{Cat}$ instead of $\mathfrak{Kel}$ — that is, by replacing L with L', defined as follows:

Definition 2.1.5 Define $L'\colon \mathrm{id}\,/\Delta \longrightarrow \mathfrak{Cat}$ to be the functor which sends an object $F\colon \mathcal{C} \longrightarrow \Delta$ of $\mathrm{id}\,/\Delta$ to $\varinjlim F$ in $\mathfrak{Cat}$, and a morphism H from $F\colon \mathcal{C} \longrightarrow \Delta$ to $G\colon \mathcal{D} \longrightarrow \Delta$ to the canonical map $\varinjlim F \longrightarrow \varinjlim G$.

Definition 2.1.6 The *categorical realisation functor* $C\colon \mathfrak{Set}^{\nabla} \longrightarrow \mathfrak{Cat}$ is the composition of the functors T from Definition 1.3.1 and L' from Definition 2.1.5.

We thus arrive at the following expanded diagram:

$$\begin{tikzcd} \mathfrak{Cat} \arrow[r, shift left, "N"] & \mathfrak{Set}^{\nabla} \arrow[l, shift left, "C"] \arrow[r, shift left, "|-|"] & \mathfrak{Kel} \arrow[l, shift left, "S"] \end{tikzcd}$$

We now prove that the pairs (C, N) and $(|-|, S)$ are adjoint.

Theorem 2.1.7 *The topological realisation functor $|-|$ is left adjoint to the singular functor S.*

Proof: According to Theorem 2 (iv) of [MAC LANE 1971], in order to prove that $|-|$ is a left adjoint it is sufficient to show that for every topological space X of $\mathfrak{Kel}$, the comma category $|-|/X$ has a terminal object.

Consider $|S(X)|$. For every object f of $\mathcal{I}_{S(X)}$, there is an object n of Δ such that $f \in S(X)_n = \mathrm{Mor}(n, X)$, so f is in fact a morphism from $n = T(S(X))(f)$ to X. By the universal property of the colimit, all such maps together induce a unique morphism ϕ from $|S(X)|$ to X. We want to show that $(S(X), \phi)$ is a terminal object of $|-|/X$.

Consider any object (Z, g) of $|-|/X$, and recall that Z and $S(X)$ are objects of $\mathfrak{Set}^{\nabla}$. Consider an object y of $\mathcal{I}_Z$, and the object $n = T(y) \in \Delta$. Then we have the map $m_y \colon n \longrightarrow Z$, courtesy of Z being the colimit of T, and moreover the map $g \circ m_y \colon n \longrightarrow X$. Now, let h be a morphism from (Z, g) to $(S(X), \phi)$, that is, a map $h \colon Z \longrightarrow S(X)$ such that $g = \phi \circ |h|$. Then for any object y of $\mathcal{I}_Z$, with $n = T_Z(y) \in \Delta$, we have $g \circ m_y = \phi \circ |h| \circ m_y$. Furthermore, by the definition of $|-|$, we have that $\phi \circ |h| \circ m_y = \phi \circ m_{T_h(y)}$. And by the definition of ϕ, we get $\phi \circ m_{T_h(y)} = h(y)$. This uniquely determines h: it has to send y to $g \circ m_y$. At the same time, this characterisation defines a morphism, so there exists a unique morphism h from (Z, g) to $(S(X), \phi)$. Hence we have shown that $(S(X), \phi)$ is a terminal object of $|-|/X$, as desired.

$$
\begin{tikzcd}
y \arrow[rr, mapsto, "T_h"] \arrow[dr, mapsto, "T_Z"'] & & T_h(y) \arrow[dl, mapsto, "T_{S(X)}"] \\
& n \arrow[dl, "m_y"'] \arrow[dr, "m_{T_h(y)}"] & \\
|Z| \arrow[rr, "|h|"] \arrow[drr, "g"'] & & |S(X)| \arrow[d, "\phi"] \\
& & X
\end{tikzcd}
.
$$

Moreover, for any topological spaces X and Y in $\mathfrak{Kel}$ and any continuous map $f \colon X \longrightarrow Y$, the following square commutes:

$$
\begin{tikzcd}
|S(X)| \arrow[r, "|S(f)|"] \arrow[d, "\phi_X"] & |S(Y)| \arrow[d, "\phi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
.
$$

Hence S is in fact the right adjoint of $|-|$. $\square$

This proof only depends on the fact that in defining $|-|$ and S, we took $\mathcal{C} = \mathcal{K}$, $\mathcal{D}_1 = \mathcal{D}_2$ and $I = J$, not on the fact that $\mathcal{C} = \mathcal{K} = \mathfrak{Kel}$ or that $\mathcal{D}_1 = \mathcal{D}_2 = \Delta$. So in particular, it also holds for the pair (C, N).

Theorem 2.1.8 *The categorical realisation functor C is left adjoint to the nerve functor N.*

Proof: Analogous to the proof of Theorem 2.1.7. $\square$

2.2 Natural transformations and homotopies

As a first, direct result, we find that the classifying space functor maps simplices in the categorical sense to simplices in the topological sense.

Proposition 2.2.1 *For every $n \in \mathbb{N}$, the classifying space of $\mathfrak{n}$ is Δ_{n-1}.*

Proof: The nerve of $\mathfrak{n}$ contains 1 non-degenerate $n-1$-simplex, it contains no higher-dimensional non-degenerate simplices and its lower dimensional simplices are exactly the faces of its $n-1$-simplex, which thus completely determines its topological realisation. $\square$

In the rest of this section, we will prove a number of important properties of the classifying space functor, most of which are due to [QUILLEN 1973]. The key property which underlies all others is the fact that the classifying space functor preserves finite limits. It is at this point that it is important that we restricted the target of the classifying space functor to the full subcategory $\mathfrak{Kel}$ of $\mathfrak{Top}$, otherwise it would not in general preserve finite limits.

Theorem 2.2.2 *The topological realisation functor $|-|\colon \mathfrak{Set}^{\nabla} \longrightarrow \mathfrak{Kel}$ commutes with finite limits.*

Proof: See [GABRIEL & ZISMAN 1967]. $\square$

Corollary 2.2.3 *The classifying space functor $|-|\colon \mathfrak{Cat} \longrightarrow \mathfrak{Kel}$ commutes with finite limits.*

Proof: This is a combination of Theorem 2.1.1 and Theorem 2.2.2. $\square$

This fact allows us to formulate a second major result, namely that under the classifying space functor, natural transformations between functors induce homotopies between continuous maps.

We want to work with analogous definitions for natural transformations and homotopies, using canonical embeddings into the product with an interval.

In the categorical setting, this interval will be the category 2, which contains two objects, X_0 and X_1. For any category $\mathcal{C}$ the embeddings which we need are $I_0\colon \mathcal{C} \xrightarrow{\sim} \{X_0\} \times \mathcal{C} \rightarrowtail 2 \times \mathcal{C}$ and $I_1\colon \mathcal{C} \xrightarrow{\sim} \{X_1\} \times \mathcal{C} \rightarrowtail 2 \times \mathcal{C}$. In the topological setting, we will use the canonical interval $\mathbb{I} = [0,1] = |2|$. For any topological space X, the embeddings which we need are $\iota_0\colon X \xrightarrow{\sim} \{0\} \times X \rightarrowtail \mathbb{I} \times X$ and $\iota_1\colon X \xrightarrow{\sim} \{1\} \times X \rightarrowtail \mathbb{I} \times X$.

Then the definitions of natural transformations and homotopies that we will appeal to are the following:

- For any categories $\mathcal{C}$ and $\mathcal{D}$ and any functors $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$, a *natural transformation* μ from F to G (or $\mu\colon F \longrightarrow G$) is a functor $2 \times \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mu \circ I_0 = F$ and $\mu \circ I_1 = G$.
- For any topological spaces X and Y and any continuous maps $f, g\colon X \longrightarrow Y$, a *homotopy* h from f to g (or $h\colon f \longrightarrow g$) is a continuous map $h\colon \mathbb{I} \times X \longrightarrow Y$ such that $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$.

Theorem 2.2.4 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F_0, F_1\colon \mathcal{C} \longrightarrow \mathcal{D}$ functors, $\mu\colon 2 \times \mathcal{C} \longrightarrow \mathcal{D}$ a natural transformation from F_0 to F_1 and $\phi\colon \mathbb{I} \times |\mathcal{C}| \xrightarrow{\sim} |2 \times \mathcal{C}|$ the canonical isomorphism from the universal property of the product. Then $|\mu| \circ \phi$ is a homotopy from $|F_0|$ to $|F_1|$.*

Proof: Let $v_0 = \phi \circ \iota_0$ and $v_1 = \phi \circ \iota_1$ be the induced embeddings $|\mathcal{C}| \rightarrowtail |2 \times \mathcal{C}|$. We have to check that for both $i = 0$ and $i = 1$, the following diagram commutes:

$$\begin{tikzcd}
|\mathcal{C}| \arrow[r, "|F_i|"] \arrow[d, tail, "v_i"'] & |\mathcal{D}| \\
|2 \times \mathcal{C}| \arrow[ur, "|\mu|"'] &
\end{tikzcd}$$

Let $I_0, I_1\colon \mathcal{C} \rightarrowtail 2 \times \mathcal{C}$ be the canonical embeddings. Since by hypothesis, $\mu \circ I_i = F_i$ and the classifying space construction is functorial, $|\mu| \circ |I_i| = |F_i|$. Hence it is sufficient to show that $v_0 = |I_0|$ and $v_1 = |I_1|$. Now let $\pi_{\mathbb{I}}$ and $\pi_{|\mathcal{C}|}$ be the projections of $|2 \times \mathcal{C}|$ onto resp. $\mathbb{I}$ and $|\mathcal{C}|$, and let P_2 and $P_{\mathcal{C}}$ be the projections of $2 \times \mathcal{C}$ onto resp. 2 and $\mathcal{C}$. Since the classifying space functor commutes with finite projective limits, $\pi_{\mathbb{I}} = |P_2|$ and $\pi_{|\mathcal{C}|} = |P_{\mathcal{C}}|$. Consider the following diagram, for $i \in \{0, 1\}$:

$$\begin{tikzcd}
& |\mathcal{C}| \arrow[d, shift right, "|I_i|"'] \arrow[d, shift left, tail, "v_i"] & \\
|\mathcal{C}| & |2 \times \mathcal{C}| \arrow[l, two heads, "\pi_{|\mathcal{C}|}"'] \arrow[r, two heads, "\pi_{\mathbb{I}}"] & \mathbb{I}
\end{tikzcd}$$

The following identities hold:

$$\pi_{|\mathcal{C}|} \circ |I_i| = |P_{\mathcal{C}}| \circ |I_i| = \mathrm{id}_{|\mathcal{C}|} = \pi_{|\mathcal{C}|} \circ v_i,$$

$$\pi_{\mathbb{I}} \circ |I_i| = |P_2| \circ |I_i| = \{0\} = \pi_{\mathbb{I}} \circ v_i.$$

By the universal property of the product, this means that $|I_i| = v_i$, which is what we wanted. ☐

The next three useful corollaries follow directly from this theorem.

Definition 2.2.5 Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and X an object of $\mathcal{D}$. Then the *constant functor* $\mathrm{Ct}_X \colon \mathcal{C} \longrightarrow \mathcal{D}$ is the functor which sends all objects of $\mathcal{C}$ to X and all morphisms to id_X.

Corollary 2.2.6 *Let $\mathcal{C}$ be a category, let X be an object of $\mathcal{C}$ and consider the identity functor* Id *and the constant functor* Ct_X *from $\mathcal{C}$ to itself. If there exists a natural transformation μ between* Id *and* Ct_X *(or vice-versa), then $|\mathcal{C}|$ is contractible.*

Proof: A constant functor factors through the category $\mathbb{1}$, so under the classifying space functor it is mapped to a continuous map which factors through Δ_0 — a constant map. According to Theorem 2.2.4 the natural transformation μ induces a homotopy between this constant map and the identity map on $|\mathcal{C}|$ (or vice-versa), which means that $|\mathcal{C}|$ is contractible. ☐

Corollary 2.2.7 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let the pair of functors (F, G) be an adjunction between $\mathcal{C}$ and $\mathcal{D}$. Then $|F|$ and $|G|$ are homotopy equivalences.*

Proof: Since F and G are adjoint, there exist natural transformation $\mu \colon FG \longrightarrow \mathrm{id}_{\mathcal{C}}$ and $\nu \colon \mathrm{id}_{\mathcal{D}} \longrightarrow GF$, which induce homotopies from $|FG|$ to $\mathrm{id}_{|\mathcal{C}|}$ and from $\mathrm{id}_{|\mathcal{D}|}$ to $|GF|$, hence $|F|$ and $|G|$ are homotopy equivalences.☐

Corollary 2.2.8 *Let $\mathcal{C}$ be a category with initial or terminal object. Then $|\mathcal{C}|$ is contractible.*

Proof: If C has an initial or terminal object, then the functor $\mathcal{C} \longrightarrow \mathbb{1}$ has an adjoint, and hence induces a homotopy equivalence between the classifying spaces $|\mathcal{C}|$ and Δ_0. ☐

2.3 The Quillen theorems

The proofs in Subsection 4.3 crucially depend on a number of well-known theorems of [QUILLEN 1973], which are repeated here without proof. Quillen's Theorem A provides us with a sufficient criterium to determine whether a functor becomes a homotopy equivalence after applying the classifying space functor. Recall Definitions 2.1.2 and 2.1.3 of the comma and cocomma categories of a functor.

Theorem 2.3.1 (Quillen A) *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. If for all $X \in \mathcal{D}$ the classifying space of the comma category F/X is contractible or for all $X \in \mathcal{D}$ the classifying space of the cocomma category $X\backslash F$ is contractible, then $|F|$ is a homotopy equivalence.*

Proof: This is Theorem A of [QUILLEN 1973]. □

Quillen's Theorem B is more general than Theorem A. For a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$, it gives conditions under which the homotopy groups of the classifying spaces of $\mathcal{C}$, $\mathcal{D}$ and the comma or cocomma categories of F form a long exact sequence. For any morphism $f\colon X \longrightarrow Y$ of $\mathcal{D}$, let $f_*\colon F/X \longrightarrow F/Y$ be the functor which sends an object $(Z, g\colon F(Z) \longrightarrow X)$ to the object $(Z, f \circ g)$, and let $f^*\colon Y\backslash F \longrightarrow X\backslash F$ be the functor which sends an object $(Z, g\colon Y \longrightarrow F(Z))$ to the object $(Z, g \circ f)$. Furthermore, for any object X of $\mathcal{D}$, let $P\colon F/X \longrightarrow \mathcal{C}$ be the functor $(Z, g) \longmapsto Z$ and $P\colon X\backslash F \longrightarrow \mathcal{C}$ the functor $(Z, g) \longmapsto Z$. Then Theorem B is:

Theorem 2.3.2 (Quillen B) *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. If for every morphism $f\colon X \longrightarrow Y$ of $\mathcal{D}$, the induced map $|f_*|\colon |F/X| \longrightarrow |F/Y|$ is a homotopy equivalence, then for every object X of $\mathcal{D}$ there exist maps $\delta_i\colon \pi_i|\mathcal{D}| \longrightarrow \pi_{i-1}|F/X|$ for all $i \geq 1$ such that*

$$\cdots \to \pi_{i+1}|\mathcal{D}| \overset{\delta_{i+1}}{\to} \pi_i|F/X| \overset{\pi_i|P|}{\to} \pi_i|\mathcal{C}| \overset{\pi_i|F|}{\to} \pi_i|\mathcal{D}| \overset{\delta_i}{\to} \pi_{i-1}|F/X| \to \cdots$$

is a long exact sequence of groups (when $i \geq 1$) and pointed sets (when $i = 0$).

If for every morphism $f\colon X \longrightarrow Y$ of $\mathcal{D}$, the induced map $|f^*|\colon |Y\backslash F| \longrightarrow |X\backslash F|$ is a homotopy equivalence, then for every object X of $\mathcal{D}$ there exist maps $\delta_i\colon \pi_i|\mathcal{D}| \longrightarrow \pi_{i-1}|X\backslash F|$ for all $i \geq 1$ such that

$$\cdots \to \pi_{i+1}|\mathcal{D}| \overset{\delta_{i+1}}{\to} \pi_i|X\backslash F| \overset{\pi_i|P|}{\to} \pi_i|\mathcal{C}| \overset{\pi_i|F|}{\to} \pi_i|\mathcal{D}| \overset{\delta_i}{\to} \pi_{i-1}|X\backslash F| \to \cdots$$

is a long exact sequence of groups (when $i \geq 1$) and pointed sets (when $i = 0$).

Proof: This is Theorem B of [QUILLEN 1973]. □

Apart from Theorems A and B, we will also make use of [QUILLEN 1973]'s Proposition 3 and its corollaries. Throughout this thesis, when considering homotopy groups and sets, the choice of a basepoint is generally irrelevant, and so will not be made explicit. Here however, it does matter, and so we make use of pointed categories and pointed topological spaces. Recall also from Subsection 1.1 that we can view a pre-ordered set as a category.

Definition 2.3.3 Let $(I, \leq)$ be a pre-ordered set. We say that I is a *directed set* if for every $x, y \in I$, there exists a $z \in I$ such that $x \leq z$ and $y \leq z$.

Proposition 2.3.4 *Let $\mathcal{I}$ be a directed set, $\mathfrak{Cat}^{\mathbf{1}}$ the category of small pointed categories and $F\colon \mathcal{I} \longrightarrow \mathfrak{Cat}^{\mathbf{1}}$ a functor. Then for any $i \in \mathbb{N}$, we have that $\pi_i|\varinjlim F| = \varinjlim(\pi_i \circ |-| \circ F)$.*

Proof: This is Proposition 3 of [QUILLEN 1973]. □

Corollary 2.3.5 *Let $\mathcal{I}$ be a directed set, $\mathfrak{Cat}^{\mathbf{1}}$ the category of small pointed categories, $F\colon \mathcal{I} \longrightarrow \mathfrak{Cat}^{\mathbf{1}}$ a functor such that for every morphism f in $\mathcal{I}$, the map $|F(f)|$ is a homotopy equivalence and $(\varinjlim F, \{m_X\}_X)$ the colimit of F. Then for every object X of $\mathcal{I}$, the map $|m_X|\colon |F(X)| \longrightarrow |\varinjlim F|$ is also a homotopy equivalence.*

Proof: By Proposition 2.3.4, $\pi_i|m_X|$ is an isomorphism for every $i \in \mathbb{N}$, so m_X is a weak homotopy equivalence. As classifying spaces have the homotopy type of a CW-complex, we can apply Whitehead's Theorem and conclude that $|m_X|$ is a strong homotopy equivalence. □

Corollary 2.3.6 *Let $\mathcal{C}$ be a directed set. Then $|\mathcal{C}|$ is contractible.*

Proof: Choose an object Z of $\mathcal{C}$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory of objects to which there is a morphism from Z. For every object X of $\mathcal{C}'$, denote by $\mathcal{C}_X \subseteq \mathcal{C}$ the full subcategory of objects of $\mathcal{C}$ from which there is a morphism to X, and consider the functor $F\colon \mathcal{C}' \longrightarrow \mathfrak{Cat}$ which sends an object X to $\mathcal{C}_X$ and a morphism to the relevant inclusion functor. These inclusion functors send the object Z (contained in every category $\mathcal{C}_X$) to itself, so we can apply Corollary 2.2.8.

As $\mathcal{C}$ is directed, for every object Y there is an object X which is greater than both Y and Z, so every morphism of $\mathcal{C}$ is contained in some $\mathcal{C}_X$ and therefore $\mathcal{C}$ is the inductive limit of F. For every category $\mathcal{C}_X$, the object X is terminal, so $|\mathcal{C}_X|$ is contractible. Hence the classifying space $|\mathcal{C}|$ is also contractible. □

3 Some monoid theory

3.1 Basic definitions

A monoid is a classical algebraic structure: a semigroup with identity element. However, a monoid can also be defined as a category with one object, and we will appeal to this second definition because it allows us to apply the concept of the classifying space in a natural way to monoids. For good order, here are the respective definitions:

Definition 3.1.1 A *monoid* is a triple $(M, \cdot_M, 1_M)$, where M is a set, $\cdot_M$ is an associative binary operation on M and 1_M is a two-sided identity element for $\cdot_M$.

Definition 3.1.2 A *monoid* is a category with one object.

The two definitions are equivalent — elements in the first, algebraic, definition correspond to morphisms in the second, categorical, definition. Composition is the binary operation on morphisms (we understand $g \circ f$ to mean the composition of f and g, *in that order*, so it corresponds to the multiplication $f \cdot g$ in the algebraic definition). Since there is only one object, all morphisms have the same source and target, so composition is defined for all pairs of morphisms. Moreover, the definition of a category requires composition to be associative and demands the existence of an identity morphism on the unique object.

From now on, we will use the two definitions of a monoid interchangeably. When working with the categorical definition of a monoid, we will denote its unique object by $*$. When using its algebraic definition, we will leave out subscripts when this should not lead to confusion, and we will usually not mention $\cdot$ and 1 explicitly, but assume their presence implicitly by just designating a set M a monoid. Also, we will usually denote the application of its binary operation through concatenation (omitting $\cdot$) and refer to it as multiplication.

Before we proceed, we note the following simple lemma:

Lemma 3.1.3 *If $(M, \cdot, 1)$ is a monoid, then 1 is the unique left and the unique right identity of M.*

Proof: Let x be an element of M. Since 1 is a two-sided identity, if x is a left identity then $x = x \cdot 1 = 1$ and if x is a right identity then $1 = 1 \cdot x = x$. □

Just as with other algebraic structures, monoid theory comes with a number of standard constructions. The ones we need are the following:

Definition 3.1.4 Let $(M, \cdot_M, 1_M)$ and $(N, \cdot_N, 1_N)$ be any monoids.

- A *monoid homomorphism from M to N* is a map from M to N that respects multiplication and sends 1_M to 1_N.
- The *product of M and N* is the monoid $(M \times N, \cdot_{M\times N}, (1, 1))$, where for any elements $(m_1, n_1), (m_2, n_2) \in M \times N$, we have that $(m_1, n_1) \cdot_{M\times N} (m_2, n_2) = (m_1 \cdot_M m_2, n_1 \cdot_N n_2)$.
- N is called a *submonoid of M* if $N \subseteq M$, if $\cdot_N$ is the restriction of $\cdot_M$ to $N \times N$ and if $1_N = 1_M$.
- For any subset $X \subseteq M$, if N is the minimal submonoid of M which contains X, we say that X *generates* N. We say that N is *finitely generated* if there exists a finite subset $Y \subseteq M$ that generates N.
- For any monoid homomorphism $f : M \longrightarrow N$, the *image of f* is the submonoid $\mathrm{im}(f) \leq N$ which contains all elements of N that have a pre-image under f.
- A *congruence* on M is an equivalence relation $\sim$ on M such that for every four elements $m_1, m_2, n_1, n_2 \in M$, with $m_1 \sim m_2$ and $n_1 \sim n_2$ we have that $m_1n_1 \sim m_2n_2$.

Monoid homomorphisms correspond exactly to functors between monoids in their categorical definition, since functors respect composition and identity morphisms. As with other algebraic structures, monoids together with monoid homomorphisms form a category: $\mathfrak{Mon}$, which is thus a full subcategory of the category $\mathfrak{Cat}$ of small categories.

By its definition, a congruence $\sim$ on a monoid M respects its monoid structure, and so the set of equivalence sets $M/\sim$ inherits this monoid structure, allowing us to view $M/\sim$ as a monoid in a natural way.

3.2 Types of monoids and examples

In this chapter, we will identify a number of different types of monoids which are relevant to our purposes and give examples to illustrate the differences. Firstly, as with any binary operation, multiplication can be commutative or not.

Definition 3.2.1 A monoid $(M, \cdot_M, 1_M)$ is said to be *commutative* if $\cdot_M$ is commutative.

Secondly, we will define groups as a type of monoid.

Definition 3.2.2 A monoid G is called a *group* if for every element x of G there exists an element $x^{-1} \in G$ (called the *inverse* of x) such that $xx^{-1} = x^{-1}x = 1$.

As monoid homomorphisms preserve inversion, a *group homomorphism* (as defined in group theory) is a monoid homomorphism between groups. This allows us to unambiguously speak simply of *homomorphisms* from now on.

Example 3.2.3 The natural numbers $(\mathbb{N}, +, 0)$ form the free monoid generated by one element. In general, the direct sum $\mathbb{N}^{(X)}$ is the free commutative monoid generated by the set X, while the free monoid generated by X consists of words built from the elements of X. (Words are multiplied through concatenation and the empty word is the unity element.)

Example 3.2.4 The trivial group $\{1\}$ is both initial and terminal in the category $\mathfrak{Mon}$.

Example 3.2.5 A monoid C is said to be cyclic if it is generated by one element $x \in C$. If C is finite, there are some minimal $k \in \mathbb{N}$ and $l \in \mathbb{N}_{>0}$ such that $x^{k+l} = x^k$, and k and l determine C up to isomorphism. If $k = 0$, then C is a cyclic group of order l. If, instead, C is infinite, there can be no such k and l, and C is isomorphic to $\mathbb{N}$. Every cyclic monoid is commutative.

The third property which we need is cancellativity:

Definition 3.2.6 A monoid M is *left-cancellative* if $zx = zy$ implies $x = y$ for every $x, y, z \in M$. It is *right-cancellative* if $xz = yz$ implies $x = y$ for every $x, y, z \in M$. It is *two-sided cancellative*, or just *cancellative*, if it is both left- and right-cancellative.

Example 3.2.7 Let $(R, +, \cdot, 0, 1)$ be a ring. Then $(R, \cdot, 1)$ forms a monoid, which is cancellative if and only if it is trivial, since $0 \cdot x = 0$ for every $x \in R$. In particular, the monoid $(\mathbb{F}_2, \cdot, 1)$ is a non-cancellative monoid with two elements.

In a cancellative monoid, inverses are unique. Cancellativity is a strictly weaker property than being a group. While all groups are cancellative, the natural numbers $\mathbb{N}$ are cancellative but don't form a group. However, for monoids that are finite the two concepts coincide:

Lemma 3.2.8 *Every cancellative finite monoid C is a group.*

Proof: Let x be any element of C. As C is cancellative, right multiplication $m \longmapsto mx$ is injective, and since C is finite, bijective, so m has a left inverse. Similarily, as left multiplication is bijective, m has a right inverse, which must coincide with its left inverse. $\square$

3.3 Cancellativisation and groupification

We are interested in cancellative monoids and groups because we can relate each monoid to a cancellative monoid and a group, and we will show in the next section for commutative monoids that this induces homotopy equivalences between classifying spaces. We make a monoid cancellative through *cancellativisation* and turn it into a group through *groupification* — the two concepts are defined analogously:

Definition 3.3.1 Let M be a monoid. A *cancellativisation of M* is a homomorphism $\phi\colon M \longrightarrow C$, with C a cancellative monoid, such that for every cancellative monoid D and every homomorphism $f\colon M \longrightarrow D$, there exists a unique homomorphism $\psi\colon C \longrightarrow D$ such that $f = \psi \circ \phi$.

Definition 3.3.2 Let M be a monoid. A *groupification of M* is a homomorphism pair $\phi\colon M \longrightarrow G$, with G a group, such that for every group H and every homomorphism $f\colon M \longrightarrow H$, there exists a unique homomorphism $\psi\colon G \longrightarrow H$ such that $f = \psi \circ \phi$.

$$\begin{tikzcd} & G \arrow[d, dashed, "! \, \psi"] \\ M \arrow[ur, "\phi"] \arrow[r, "f"'] & H \end{tikzcd}$$

By their nature, cancellativisation and groupification are unique up to a unique isomorphism. Due to the fact that groups are cancellative, we have the following lemma:

Lemma 3.3.3 *Let M be a monoid, $\phi\colon M \longrightarrow C$ a cancellativisation of M and $\phi'\colon C \longrightarrow G$ a groupification of C. Then $\phi'\phi\colon M \longrightarrow G$ is a groupification of M.*

Proof: Due to the conditions satisfied by cancellativisation and groupification, for every group H and every homomorphism $f\colon M \longrightarrow H$, there exist unique morphisms $\psi\colon C \longrightarrow H$ and $\psi'\colon G \longrightarrow H$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[rr, "\phi'"] \arrow[dr, dashed, "\psi", "!"'] & & G \arrow[dl, dashed, "\psi'"', "!"] \\ M \arrow[ur, "\phi"] \arrow[rr, "f"'] & & H & \end{tikzcd}$$

□

We now show that the cancellativisation and groupification of a monoid actually exist.

Proposition 3.3.4 *Let M be a monoid, and let $\sim$ be the minimal congruence which contains (x, y) for every $x, y \in M$ for which there exists a $z \in M$ with $xz = yz$ or $zx = zy$. Then the homomorphism $\phi\colon M \twoheadrightarrow M/\sim$ defined by $x \longmapsto [x]$ is the cancellativisation of M.*

Proof: If C is any cancellative monoid, and $f\colon M \longrightarrow C$ any homomorphism, then the condition that any $[x] \in M/\sim$ be sent to $f(x)$ defines a unique homomorphism from $M/\sim$ to C. □

For groupification, we take a *free group* and divide out by a congruence, following the general strategy of e.g. Construction 12.3 of [CLIFFORD & PRESTON 1967]). The free group generated by a set X is the free monoid generated by X and a copy X' of X which contains a symbol x^{-1} for every $x \in X$, modulo the minimal congruence which for any $x \in X$ identifies the words xx^{-1} and $x^{-1}x$ with the empty string.

Proposition 3.3.5 *Let M be a monoid, let F be the free group generated by the underlying set of M and let $\sim$ be the minimal congruence on F containing (xy, z) for all $x, y, z \in M$ such that $xy = z$. Then the homomorphism $\phi\colon M \longrightarrow F/\sim$ defined by $x \longmapsto [x]$ is the groupification of M.*

Proof: Let G be any group, and $f\colon M \longrightarrow G$ any homomorphism. Then there exists a unique homomorphism from F to G which for any $x \in M$ sends the word x to $f(x)$. Like $\sim$, this homomorphism equates the words xy and z for any $x, y, z \in M$ such that $xy = z$, so it factors through $F/\sim$ and we obtain a unique homomorphism $\psi\colon F/\sim \longrightarrow G$ such that $f = \psi \circ \phi$. □

From now on, we will keep in mind these particular constructions for cancellativisation and groupification. We will also make use of the following fact:

Lemma 3.3.6 *The groupification $\phi\colon C \longrightarrow F/\sim$ of a commutative cancellative monoid C is injective.*

Proof: As C is commutative, any element of $F/\sim$ is of the form $[xy^{-1}]$ for some $x, y \in C$, with $[x_1y_1^{-1}][x_2y_2^{-1}] = [(x_1 \cdot x_2)(y_1 \cdot y_2)^{-1}]$. Since for any $x \in C$, the concatenation xx^{-1} is the empty word in F, we find that the elements $[x_1y_1^{-1}]$ and $[x_2y_2^{-1}]$ of $F/\sim$ are the same if $x_1y_2 = x_2y_1$. I claim that this fully characterises $F/\sim$, since it defines a congruence on F. The relation is clearly reflexive and symmetric. It is transitive (and hence an equivalence relation) for if $x_1y_2 = x_2y_1$ and $x_2y_3 = x_3y_2$, then $x_1x_2y_3 = x_1x_3y_2 = x_2x_3y_1$, so since C is cancellative, $x_1y_3 = x_3y_1$. It is a congruence, for if $x_1y_2 = x_2y_1$ and $x_3y_4 = x_4y_3$, then we also have $x_1x_3y_2y_4 = x_2x_4y_1y_3$.

Now, if $x_1, x_2 \in C$ are distinct elements, $[x_1]$ and $[x_2]$ are not equated by $\sim$, so $x \longmapsto [x]$ is injective. □

It should be noted that the previous lemma does not in general hold for non-commutative monoids. A counterexample was first given in [MALCEV 1937], it is repeated below.

Example 3.3.7 Consider the free monoid $F = \langle a, b, c, d, x, y, u, v \rangle$, and let $\sim$ be the congruence on F generated by the relation containing (ax, by), (cx, dy) and (au, bv). It is shown in [MALCEV 1937] that $\sim$ does not equate cu and dv and that the monoid $F/\sim$ is cancellative. But it is not embeddable in a group, since if it were, we would have $[d^{-1}c] = [yx^{-1}] = [b^{-1}a] = [vu^{-1}]$, hence $[cu] = [dv]$.

Corollary 3.3.8 *Let M be a commutative monoid and $\phi\colon M \longrightarrow G$ its groupification. Then the induced map $M \twoheadrightarrow \mathrm{im}(\phi)$ is its cancellativisation.*

Proof: By Lemma 3.3.3, ϕ is the composition of the cancellativisation $\eta\colon M \twoheadrightarrow C$ of M and the groupification $\iota\colon C \rightarrowtail G$ of C. It was proven in 3.3.6 that ι is injective, and η is surjective by design, so C is the image of ϕ. $\square$

3.4 The cocomma category of a monoid homomorphism

In Subsection 4.3, we will prove that the classifying space functor turns the groupification map of a commutative monoid into a homotopy equivalence. To prove this, we will appeal to Theorem 2.3.1, according to which a functor F induces a homotopy equivalence if the classifying spaces of all its comma categories F/X or of all its cocomma categories $X\backslash F$ are contractible (for all objects X of the target of F). A monoid has but one object, so if f is a homomorphism of monoids, we can speak of *the* comma category $f/*$ and *the* cocomma category $*\backslash f$. Both admit a straightforward algebraic description:

Lemma 3.4.1 *Let M and N be monoids, and $f\colon M \longrightarrow N$ a homomorphism. Then the comma category $f/*$ is isomorphic to the following category. Its objects are the elements of N. A morphism with source s and target t is a triple (m, s, t), with $m \in M$ such that $s = f(m)t$. The composition of any morphisms (m_1, s, t) and (m_2, t, u) is (m_2m_1, s, u).*

The cocomma category $*\backslash f$ is isomorphic to the following category. Its objects are also the elements of N, but its morphisms are triples (m, s, t) such that $sf(m) = t$, and the composition of morphisms (m_1, s, t) and (m_2, t, u) is (m_1m_2, s, u).

I find cocomma categories intuitively more appealing, and will therefore make use of them wherever possible. We will usually represent a morphism (m, s, t) as $m\colon s \longrightarrow t$, even though m on its own does not necessarily determine a unique morphism.

There are a couple more things which we can say about the cocomma category $*\backslash f$ of a monoid homomorphism $f\colon M \longrightarrow N$. A morphism $m\colon s \longrightarrow t$ of $*\backslash f$ is an isomorphism if and only if m is a unit of M. If f is injective and N is cancellative, the morphisms of $*\backslash f$ are uniquely determined by their source and target. The set of morphisms thus forms a binary relation on the set of objects of $*\backslash f$. Due to the fact that morphisms can be composed and the existence of identity morphisms, this relation is transitive and reflexive, hence a pre-order. It is furthermore antisymmetric ($(x \leq y) \wedge (y \leq x) \implies x = y$) and hence a partial order if and only if M contains no non-trivial units. For example, if $\iota\colon \mathbb{N} \rightarrowtail \mathbb{Z}$ is the canonical embedding, the cocomma category $*\backslash\iota$ is just $\mathbb{Z}$ equipped with the canonical order $\leq$.

4 The classifying space of a monoid

In this section, we will determine as much as we can about the classifying space of a monoid. Subsection 4.1 will cover the classifying space of a group. We will build on this in Subsection 4.2 by establishing for any monoid the fundamental group of its classifying space. Finally, we will prove that the groupification $\phi\colon M \longrightarrow G$ of a monoid M induces a homotopy equivalence between classifying spaces if the monoid is commutative (Subsection 4.3) or a free monoid (Subsection 4.4).

4.1 The classifying space of a group

In this subsection, we will prove that the classifying space of a group G is a so-called Eilenberg-Maclane space of type $(G, 1)$, that is a connnected space whose fundamental group is isomorphic to G and whose other homotopy groups are trivial. This tells us quite a lot about the space, since we have the following general result:

Theorem 4.1.1 *For any group G, all Eilenberg-MacLane spaces of type $K(G, 1)$ are homotopy equivalent.*

Proof: See Theorem 1.B.8 of [HATCHER 2002]. □

That the classifying space of a group has trivial higher homotopy groups follows rather nicely from Quillen's Theorem B:

Lemma 4.1.2 *Let G be a group. Then for every $i \geq 2$, the homotopy group $\pi_i|G|$ is trivial.*

Proof: Consider the trivial group 0, which is isomorphic to the category $\mathbb{1}$ and whose classifying space is a point. The cocomma category $*\backslash\iota$ of the canonical map $\iota\colon 0 \rightarrowtail G$ is a discrete set, and its classifying space therefore a discrete space. Any element g of G is invertible, so the induced map $g_*\colon *\backslash\iota \longrightarrow *\backslash\iota$ is an isomorphism, which induces in turn a homotopy equivalence between classifying spaces, so the conditions of Theorem 2.3.2 are satisfied and we arrive at the following long exact sequence:

$$\cdots \longrightarrow \pi_{i+1}|G| \xrightarrow{\delta_{i+1}} \pi_i|*\backslash\iota| \xrightarrow{\pi_i|P|} \pi_i|0| \xrightarrow{\pi_i|\iota|} \pi_i|G| \xrightarrow{\delta_i} \pi_{i-1}|*\backslash\iota| \longrightarrow \cdots$$

Since $\pi_i|0|$ and $\pi_i|*\backslash\iota|$ are trivial for all $i \geq 1$, so is $\pi_i|G|$ for all $i \geq 2$. □

The previous lemma also provides us with the bijection δ_1 between $\pi_1|G|$ and the set $\pi_0|*\backslash\iota|$, which can be identified with the underlying set of G. This strongly suggests that $\pi_1|G|$ and G are actually isomorphic as groups, and closer inspection of $\delta_1\colon \pi_1|G| \longrightarrow G$ will reveal that it is in fact a homomorphism. Proving this, however, would require us to delve deep in the proof of Quillen's Theorem B. So instead, we will replicate below [HATCHER 2002]'s proof of the fact that $|G|$ is an Eilenberg-MacLane space of type $K(G,1)$. An additional motivation is that we will require elements of this proof in the next subsection to show that the fundamental group of the classifying space of any monoid is its groupification.

The strategy of [HATCHER 2002]'s proof is first to construct from a group G a space $E(G)$ and an action of G on $E(G)$. We then show that the quotient of $E(G)$ under this group action is homeomorphic to $|G|$ and prove that the quotient map is a universal cover of $|G|$. It then follows that the fundamental group of $|G|$ is isomorphic to the deck-transformation group of $(E(G), \phi)$, which by construction is G, and the other homotopy groups of $|G|$ are isomorphic to the homotopy groups of $E(G)$, which turn out to be trivial.

The following definitions and lemma stem from Example 1.B.7 of [HATCHER 2002].

Definition 4.1.3 Let G be a group. Then define $X(G)$ to be the following simplicial set. For every $n \in \mathbb{N}$, X_{n} is the set of n-tuples in G. For every $1 \leq i \leq n$, the generating face map d_i omits the i^{th} co-ordinate and the generating degeneracy map s_i duplicates the i^{th} co-ordinate. Set $E(G) = |X(G)|$ and for any $g_1, g_2, \ldots, g_n \in G$ write $[g_1, g_2, \ldots, g_n]$ for the image of Δ_{n-1} in $E(G)$ induced by $(g_1, g_2, \ldots, g_n)$.

Definition 4.1.4 Let G be a group. Then $\xi \colon G \longrightarrow \mathrm{Aut}(E(G))$ is the following group action. For any $g \in G$, let ξ_g send an n-simplex $h = [h_1, h_2, \ldots, h_n]$ to $gh = [gh_1, gh_2, \ldots, gh_n]$.

The action ξ is well defined since it respects face and degeneracy maps.

Lemma 4.1.5 *$E(G)/\xi$ is homeomorphic to $|G|$.*

Proof: Consider an n-simplex $X = [g_1, g_2, \ldots, g_n]$ in $E(G)$. Since G is a group, X can also be written as

$$[g_1, g_1h_1, \ldots, g_1h_1 \ldots h_{n-1}],$$

with $h_i = g_i^{-1}g_{i+1}$ for $1 \leq i \leq n-1$. The quotient map identifies X exactly with all other n-simplices of the form

$$[g_1', g_1'h_1, \ldots, g_1'h_1 \ldots h_{n-1}],$$

for any $g_1' \in G$. Hence, the resulting equivalence class can be uniquely identified as

$$[h_1|h_2| \ldots |h_{n-1}].$$

We define a homeomorphism from $E(G)/\phi$ to $|G|$ by sending $[h_1|h_2| \ldots |h_{n-1}]$ homeomorphically to the image of Δ_{n-1} in $|G|$ induced by the $n-1$-simplex $(h_1, h_2, \dots, h_{n-1})$ of $N(G)$. This is well defined since the i^{th} face of

$$[h_1|h_2|\dots|h_{i-1}|h_i|\dots|h_{n-1}]$$

is

$$[h_1|h_2|\dots|h_{i-1}h_i|\dots|h_{n-1}]$$

and its i^{th} degeneracy is

$$[h_1|h_2|\dots|h_{i-1}|1|h_i|\dots|h_{n-1}].$$ □

We will now prove that the quotient map induced by the group action ξ is a covering map, combining Proposition 1.40, the text preceding it on page 72 and Exercise 1.B.1 of [HATCHER 2002].

Lemma 4.1.6 *The quotient map $q\colon E(G) \twoheadrightarrow E(G)/\xi$ is a covering map.*

Proof: We have to prove that every point of $E(G)/\xi$ has a neighbourhood U such that its pre-image under q is a disjoint union of sets which are mapped homeomorphically onto U by q.

Let x be a point of $E(G)$. Then there exist simplices of $E(G)$ that contain x, and we can choose such an n-simplex X with n minimal. Note that X is not degenerate and that if X is not 0-dimensional, x is contained in the interior of X, for else x would be contained in a simplex of yet lower dimension. Furthermore, X is the unique n-simplex that contains x, for two non-degenerate n-simplices can only possibly intersect in (parts of) their boundaries. On the other hand, for any $g \in G\backslash\{1\}$, the map ξ_g sends X to an n-simplex different from itself. In particular, $\xi_g(x) \neq x$ (the action ξ is free). Hence we can choose an open neighbourhood U_x of x such that for every m-simplex Y that contains x, the set U_x contains no points of Y which lie closer to another n-dimensional face of Y than to X (using the metric of $\mathbb{R}^{m+1}$). Since ξ_g maps simplices linearly onto each other, this means that $\xi_g[U_x] \cap U_x = \emptyset$. It follows that for any two distinct $g_1, g_2 \in G$, the two spaces $\xi_{g_1}[U_x]$ and $\xi_{g_2}[U_x]$ are disjoint, for otherwise $\xi_{g_1 g_1^{-1}}[U_x] = U_x$ and $\xi_{g_2 g_1^{-1}}[U_x]$ would not be disjoint either. We can thus take $q[U_x]$ to be our desired neigbourhood of $q(x)$, which is open since its pre-image, the disjoint union $\bigcup_{g \in G} \xi_g[U_x]$ is open. By construction, for every $g \in G$ the restriction $q|_{\xi_g[U_x]}$ is a homeomorphism. □

Corollary 4.1.7 *The composition $\phi\colon E(G) \twoheadrightarrow E(G)/\xi \xrightarrow{\sim} |G|$ is a covering map.*

Next, we will prove that $E(G)$ is contractible, which is again adapted from Example 1.B.7 of [HATCHER 2002]. This fact implies that the covering map

$E(G) \longrightarrow\!\!\!\!\rightarrow |G|$ is universal, and it also provides us with a second proof that the higher homotopy groups of $|G|$ are trivial.

Lemma 4.1.8 *For any group G, the space $E(G)$ is contractible.*

Proof: Let $[1]$ be the realisation of the identity element of G in $E(G)$. We will construct a homotopy h from the identity map to the constant map onto $[1]$. Let $x \in E(G)$ and let $[g_1, g_2, \ldots, g_n]$ be the realisation of a non-degenerate n-simplex containing x, for some vertices $g_1, g_2, \ldots, g_n \in G$, one of which equals 1. The pre-images of x and $[1]$ in Δ_{n-1} are connected by a unique line segment, which is realised as a path from x to $[1]$; define h by having it transform x into $[1]$ linearly along this path. This definition does not depend on the choice of simplex containing x. $\square$

Corollary 4.1.9 *The pair $(E(G), \phi)$ is a universal cover of G.*

Proof: Since $E(G)$ is contractible, it is simply connected, and simply connected covers are universal. $\square$

Theorem 4.1.10 *Let G be a group. Then $|G|$ is an Eilenberg-MacLane space of type $K(G, 1)$.*

Proof: Because the action $\xi\colon G \longrightarrow \mathrm{Aut}(E(G))$ from Definition 4.1.4 is injective, G is the deck transformation group of the cover $(E(G), \phi)$ of $|G|$. By Corollary 4.1.9, $(E(G), \phi)$ is a universal cover, so G is isomorphic to $\pi_1|G|$. The other homotopy groups of $|G|$ are trivial because $E(G)$ is contractible. $\square$

With this result, we can say the following about the long exact sequence of Theorem 2.3.2:

Corollary 4.1.11 *Let $f\colon G \longrightarrow H$ be a morphism of groups. Then the long exact sequence of Theorem 2.3.2 for cocomma categories reduces to the following: (Keeping in mind that $\pi_0|*\backslash f| = \operatorname{coker} f$ is in general only a set, not a group.)*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_1|*\backslash f| & \longrightarrow & \pi_1|G| & \longrightarrow & \pi_1|H| & \longrightarrow & \pi_0|*\backslash f| & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & \ker f & \rightarrowtail & G & \xrightarrow{f} & H & \twoheadrightarrow & \operatorname{coker} f & \longrightarrow & \{0\}
\end{array}$$

Proof: Two objects x and y of $*\backslash f$ are in the same connected component if and only if as elements of H they are part of the same equivalence class of $\operatorname{coker} f$. As G is a group, all morphisms of the cocomma category $*\backslash f$ are isomorphisms, so every connected component of $*\backslash f$ is equivalent (in the categorical sense) to the full subgroup $\mathrm{Aut}(x)$ for any of its objects x, the kernel of f. $\square$

4.2 The fundamental group of a monoid

In this subsection, we will show that if M is any monoid, and $\phi \colon M \longrightarrow G$ its groupification, then $\pi_1|\phi|$ is an isomorphism. We construct a homomorphism ϕ' from M to the fundamental group $\pi_1|M|$ of its classifying space, and prove that ϕ' satisfies the universal property of groupification.

Definition 4.2.1 Let M be a monoid. Define the map $\phi' \colon M \longrightarrow \pi_1|M|$ as follows. Any element $m \in M$ gives rise to a 1-simplex in its nerve. This 1-simplex is realised as a path f_m in $|M|$ which starts and ends in the realisation of the unique 0-simplex, induced by the unique object $*$ of M. Hence f_m is a loop, and we set $\phi'(m) = [f_m]$ in $\pi_1|M|$.

Lemma 4.2.2 *The map ϕ' is a homomorphism.*

Proof: Let $m, n \in M$. In the nerve of M, (m), (n) and (mn) are the faces of the 2-simplex (m, n), so in the realisation $|M|$, the loops f_m, f_n and f_{mn} are the three sides of an image of Δ_2. Hence we can homotopically transform $f_n \circ f_m$ into f_{mn}, so $[f_n \circ f_m] = [f_{mn}]$. Furthermore, (1) is degenerate and realised as the constant loop, so $\phi'(1)$ is the identity element of $\pi_1|M|$ as required. □

For the following lemma, recall the covering space $(E(G), \phi)$ of a group G from Subsection 4.1.

Lemma 4.2.3 *Let G be a group. Then $\phi' \colon G \longrightarrow \pi_1|G|$ is an isomorphism.*

Proof: Consider the covering map $\phi \colon E(G) \longrightarrow |G|$ and call x_* the point in $|G|$ induced by the unique object $*$ of G and x_m the point in $E(G)$ induced by (m), for any $m \in M$. Then $\phi^{-1}[x_*] = \{x_m | m \in M\}$. The homotopy lifting property (proved e.g. as Proposition 1.30 in [HATCHER 2002]) tells us two things. First, that of any loop in $|G|$, there exists a unique lift (a path in $E(G)$) starting in x_1 (and necessarily ending in x_m for some $m \in M$). Second, that of any homotopy between any two loops in $|G|$, there exists a unique lift starting in any given lift of the first loop. This means that ϕ' is injective. For if it were not, there would exist a homotopy between the two loops induced by $(1, m)$ and $(1, n)$, for some distinct $m, n \in M$, constant on start and end points. But given the existence and relative uniqueness of lifts just appealed to, this homotopy would lift to a homotopy between a path running from x_1 to x_m and a path running from x_1 and x_n, which couldn't possibly keep endpoints in the pre-image of x_*.

The homomorphism ϕ' is also surjective. Any loop in f in $|G|$ lifts to a path in $E(G)$ starting in x_1 and ending in some x_m. As $E(G)$ is contractible by Lemma 4.1.8, there exists a homotopy between this path and the path induced by $(1, m)$, constant on start and end points. This homotopy is mapped by ϕ to a homotopy between f and the loop induced by $(1, m)$, which means that $[f] = \phi'(m)$. □

Lemma 4.2.4 *For all monoids M_1, M_2 and every morphism $f\colon M_1 \longrightarrow M_2$ the following square commutes:*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\phi'"'] & M_2 \arrow[d, "\phi'"] \\ \pi_1|M_1| \arrow[r, "\pi_1|f|"] & \pi_1|M_2| \end{tikzcd}$$

Proof: This follows from the definitions of the functors N, $|-|$ and π_1. Any element m of M_1 is a 1-simplex in the nerve of M_1 which is mapped by $N(f)$ to the 1-simplex corresponding to $f(m)$ in $N(M_2)$. Subsequently, m becomes a loop in $|M_1|$ which is mapped by $|f|$ to the loop corresponding to $f(n)$ in $|M_2|$. Finally, after application of π_1, this loop is a representative of the homotopy class $\phi'(m)$ in $\pi_1|M_1|$, which is mapped by $\pi_1|f|$ to the homotopy class $\phi'(f(m))$ in $\pi_1|M_2|$ of which the loop corresponding to n is a representative. □

Remark 4.2.5 The previous lemma in fact says that $\{\phi'_M\}_M$ is a natural transformation between the identity functor on $\mathfrak{Mon}$ and the composition $\pi_1 \circ |-|$.

Lemma 4.2.6 *For any monoid M, the image of $\phi' : M \longrightarrow \pi_1|M|$ generates $\pi_1|M|$ as a group.*

Proof: Let $f : \mathbb{I} \longrightarrow |M|$ be any loop in $|M|$. As $\mathbb{I}$ is compact, f is continuous and $|M|$ is a CW complex, the image of f is also compact. Hence by Proposition A.1 of [HATCHER 2002], the image of f sits inside a finite subcomplex of $|M|$, of dimension n for some $n \in \mathbb{N}$. If $n \geq 2$, then any n-simplex of $|M|$ has a boundary of $n-1$-simplices, so we can homotopically transform f into a loop whose image only intersects a finite $n-1$-dimensional subcomplex of $|M|$. Repeating this procedure, we find that f is homotopic to a loop whose image is entirely contained in a finite, 1-dimensional subcomplex of $|M|$, and thus, to a loop that traces a series of 1-simplices in normal and reverse directions. So f is a representative of the equivalence class $\phi'(m_1)^{j_1}\phi'(m_2)^{j_2}\ldots\phi'(m_k)^{j_k}$ of $\pi_1|M|$ for some $k \in \mathbb{N}$, some $m_1, m_2, \ldots, m_k \in M$ and $j_1, j_2, \ldots, j_k \in \{1, -1\}$. □

Theorem 4.2.7 *Let M be a monoid, and let $\phi\colon M \longrightarrow G$ be its groupification. Then ϕ induces an isomorphism between $\pi_1|M|$ and $\pi_1|G|$.*

Proof: Let H be any group, and let $f\colon M \longrightarrow H$ be any monoid homomorphism. By Lemmata 4.2.3 and 4.2.4 we can construct the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "\phi"] \arrow[dd, "\phi'"'] & & H \arrow[dd, "\phi'", "\wr"'] \\
& G \arrow[ur, "\psi_f"] \arrow[dl, "\psi_{\phi'}"] & \\
\pi_1|M| \arrow[rr, "\pi_1|f|"] & & \pi_1|H|
\end{tikzcd}$$

We see that $f = (\phi'^{-1} \circ \pi_1|f|) \circ \phi'$. By the universal property of groupification, any g such that $f = g \circ \phi'$ also satisfies $\psi_f = g \circ \psi_{\phi'}$, but by Lemma 4.2.6, $\psi_{\phi'}$ is surjective, so g has to equal $(\phi'^{-1} \circ \pi_1|f|)$. Hence $\phi'\colon M \longrightarrow \pi_1|M|$ satisfies the universal property of groupification, so $\pi_1|M|$ is G up to a unique isomorphism.

If we now apply the functor $\pi \circ |-|$ to the groupification map ϕ, we obtain the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "\phi'"'] & G \arrow[d, "\phi'", "\wr"'] \\
\pi_1|M| \arrow[r, "\pi_1|\phi|"] & \pi_1|G|
\end{tikzcd}$$

By the universal property of groupification, $\pi_1|\phi|$ has to be an isomorphism. $\square$

4.3 The classifying space of a commutative monoid

The aim of this section is to prove the following theorem:

Theorem 4.3.1 *Let M be a commutative monoid and let $\phi\colon M \longrightarrow G$ be its groupification. Then $|\phi|$ is a homotopy equivalence.*

The map ϕ factors into a surjection η followed by an injection ι:

$$\begin{tikzcd}
M \arrow[rr, "\phi"] \arrow[dr, two heads, "\eta"'] & & G \\
& \mathrm{im}(\phi) \arrow[ur, tail, "\iota"'] &
\end{tikzcd}$$

By Corollary 3.3.8, η is the cancellativisation of M and ι the groupification of $\mathrm{im}(\phi)$. We will prove in Proposition 4.3.3 that $|\eta|$ and in Proposition 4.3.2 that $|\iota|$ is a homotopy equivalence.

It follows almost straight from Quillen's theorems that $|\iota|$ is a homotopy equivalence:

Proposition 4.3.2 *Let C be a cancellative commutative monoid and let $\iota\colon C \rightarrowtail G$ be its groupification. Then $|\iota|$ is a homotopy equivalence.*

Proof: By Theorem 2.3.1, it is sufficient to show that the classifying space of the cocomma category $*\backslash\iota$ is contractible. Since ι is injective and G cancellative, $*\backslash\iota$ is a pre-ordered set. Since C is cancellative and commutative, any element of G can be written as $[cd^{-1}]$ for some $c, d \in C$ (given the construction of G in Proposition 3.3.5). So for any elements $[c_1 d_1^{-1}], [c_2 d_2^{-1}]$, we have morphisms $d_1 c_2\colon c_1 d_1^{-1} \longrightarrow c_1 c_2$ and $d_2 c_1\colon c_2 d_2^{-1} \longrightarrow c_1 c_2$ to a common third element. Hence $*\backslash\iota$ is directed, and its classifying space therefore contractible by Corollary 2.3.6. □

To prove that $|\eta|$ is a homotopy equivalence, we show that the cancellativisation of a monoid M is the limit of a directed inductive system containing M and in which all morphisms induce homotopy equivalences between classifying spaces. The proposition requires two lemmata which are proven afterwards.

Proposition 4.3.3 *Let M be a commutative monoid and $\eta\colon M \twoheadrightarrow C$ its cancellativisation. Then $|\eta|$ is a homotopy equivalence.*

Proof: For every $x \in M$, let $\sim_x$ be the congruence which equates $m_1, m_2 \in M$ if $m_1 x = m_2 x$. We can order the set of congruences $\mathcal{I} = \{\sim_x | x \in M\}$ by inclusion. For any $x_1, x_2 \in M$, both $\sim_{x_1}$ and $\sim_{x_2}$ are included in $\sim_{x_1 x_2}$, so $\mathcal{I}$ is directed. Consider the functor $F\colon \mathcal{I} \longrightarrow \mathfrak{Mon}$ which sends $\sim_x$ to the quotient monoid $M/{\sim_x}$, and an inclusion map $\sim_x \rightarrowtail \sim_y$ to the induced quotient map $M/{\sim_x} \twoheadrightarrow M/{\sim_y}$. The cancellativisation C of M was constructed in Proposition 3.3.4 as the quotient of the relation which equates any elements m_1 and m_2 if there is a third element z such that $zm_1 = zm_2$. So C is the inductive limit of F. By Lemma 4.3.5 below, for every $x \in M$ the quotient map $M \twoheadrightarrow M/{\sim_x}$ induces a homotopy equivalence between classifying spaces. It follows that all inclusion maps in $\mathcal{I}$ induce homotopy equivalences, hence by Corollary 2.3.5 the cancellativisation map $\eta\colon M \twoheadrightarrow C$ also induces a homotopy equivalence. □

To prove that for any $x \in M$, the quotient map $q\colon M \longrightarrow\!\!\!\!\rightarrow M/{\sim_x}$ induces a homotopy equivalence, we proceed as follows. If Id is the identity functor on the cocomma category $*\backslash q$, and Ct the constant functor to 1, we want to show that $|\mathrm{Id}|$ and $|\mathrm{Ct}|$ are homotopic. We do this by considering a third functor F_x and provide homotopies from both $|\mathrm{Id}|$ and $|\mathrm{Ct}|$ to $|F_x|$.

Lemma 4.3.4 *Let M and N be commutative monoids, let $f\colon M \longrightarrow N$ be a homomorphism, and for every $x \in M$, let $F_x\colon *\backslash f \longrightarrow *\backslash f$ be the functor which sends an object n_1 to the object $n_1 f(x)$ and a morphism $m\colon n_1 \longrightarrow n_2$ to the morphism $m\colon n_1 f(x) \longrightarrow n_2 f(x)$. Then for every $x \in M$, there exists a homotopy from the identity map $|\mathrm{Id}|$ to $|F_x|$.*

Proof: For any $n_1, n_2 \in N$ and any $m \in M$ such that $n_1 f(m) = n_2$, the following square commutes in the cocomma category $*\backslash f$:

$$\begin{tikzcd}
n_1 \arrow[r, "m"] \arrow[d, "x"'] & n_2 \arrow[d, "x"] \\
n_1 f(x) \arrow[r, "m"] & n_2 f(x)
\end{tikzcd}$$

Hence $\{x\}_n$ is a natural transformation from the identity functor on $*\backslash f$ to F_x, and by Theorem 2.2.4 this natural transformation induces a homotopy from $|\mathrm{Id}|$ to $|F_x|$. $\square$

Lemma 4.3.5 *Let M be a commutative monoid, let x be an element of M, let $\sim$ be the congruence which equates $m_1, m_2 \in M$ if $m_1 x = m_2 x$ and let $q\colon M \longrightarrow\!\!\!\!\rightarrow M/{\sim}$ be the quotient map. Then $|q|$ is a homotopy equivalence.*

Proof: Our aim is to prove that the classifying space of the cocomma category $*\backslash q$ is contractible, which implies by Theorem 2.3.1 that $|q|$ is a homotopy equivalence. It was shown in Lemma 4.3.4 that the identity map $|\mathrm{Id}|$ is homotopic to $|F_x|$. We will now show that also the map $|\mathrm{Ct}|$ induced by the constant functor on 1 is homotopic to $|F_x|$, thus establishing the contractibility of $*\backslash q$ by Corollary 2.2.6.

An object of $M/{\sim}$ is of the form $q(m)$ for some $m \in M$, and is sent to 1 by G and to $q(mx)$ by the functor F_x. For any morphism $y\colon m_1 \longrightarrow m_2 \in M$, consider the following square in $*\backslash q$.

$$\begin{tikzcd}
1 \arrow[r, "1"] \arrow[d, "m_1 x"'] & 1 \arrow[d, "m_2 x"] \\
q(m_1 x) \arrow[r, "y"] & q(m_2 x)
\end{tikzcd}$$

We claim $\{mx\}_{q(m)}$ is a natural transformation from Ct to F_x. The morphisms mx uniquely depend on $q(m)$ since by definition, $q(m') = q(m)$ if and only if $mx = m'x$. Since y is a morphism from $q(m_1)$ to $q(m_2)$, we know that $q(m_1 y) = q(m_2)$, so the diagram commutes.

So $\{mx\}_{q(m)}$ is a natural transformation from Ct_1 to F_x and by Theorem 2.2.4 it induces a homotopy from $|\mathrm{Ct}|$ to $|F_x|$. $\square$

4.4 The classifying space of a free monoid

In this section, the following proposition will be proved:

Proposition 4.4.1 *Let S be a set, M a free monoid generated by S, and $\phi\colon M \rightarrowtail G$ its groupification. Then $|\phi|$ is a homotopy equivalence.*

For the rest of this subsection, we will keep ϕ fixed. The idea of the proof is to formalise the intuition that, if we represent the cocomma category $*\backslash\phi$ as a graph, it does not have any cycles and thus certainly *looks* contractible. Note that since ϕ is injective and M does not contain any units, $*\backslash\phi$ is a partially ordered set. We will call a morphism of $*\backslash\phi$ *generating* if it corresponds to a generating element of M.

A subcategory of $*\backslash\phi$ may have 'protruding' objects, which correspond to what are called leafs in graph theory:

Definition 4.4.2 Let $\mathcal{C}$ be a subcategory of $*\backslash\phi$. A generating morphism $f\colon X \longrightarrow Y$ of $\mathcal{C}$ is called an *initial leaf* if X is not the target of any non-trivial morphism of $\mathcal{C}$, and if every morphism whose source is X factors through f. It is called a *terminal leaf* if Y is not the source of any non-trivial morphism of $\mathcal{C}$ and if every morphism whose target is Y factors through f.

'Folding in' a leaf does not affect the homotopy type of the classifying space of a subcategory $\mathcal{C}$ of $*\backslash\phi$:

Lemma 4.4.3 *Let $\mathcal{C}$ be a subcategory of $*\backslash\phi$ with a terminal leaf $\omega\colon Y \longrightarrow Z$ and let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ that contains all its objects save Z and all its morphisms except those whose target is Z. Let $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ be the functor which sends Z to Y, any morphism f whose target is Z to the unique morphism f' for which $\omega f' = f$ and which is the identity on all other objects and morphisms of $\mathcal{C}$. Then $|F|$ is a homotopy equivalence.*

Proof: For any object X of $\mathcal{C}'$, consider the cocomma category $X\backslash F$. We will show that (X, id) is an initial object of $X\backslash F$. Let (W, f) be any object of $X\backslash F$. If W is any object of $\mathcal{C}$ other than Z, then $F^{-1}(f) = \{f\}$, so $f\colon X \longrightarrow W$ is the unique morphism to make the diagram commute. If $W = Z$, by assumption

ωf is the unique morphism from X to Z which is mapped to f, thus making the diagram commute.

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}"] \arrow[dr, "f"] & \\
F(X) = X \arrow[r, dashed] & F(W)
\end{tikzcd}$$

Since $X \backslash F$ has an initial object, its classifying space is contractible by Corollary 2.2.8. Since for every object X of $\mathcal{C}'$, the classifying space of the co-comma category $X \backslash F$ is contractible, $|F|$ is a homotopy equivalence by Theorem 2.3.1. $\square$

Lemma 4.4.4 *Let $\mathcal{C}$ be a subcategory of $* \backslash \phi$ with an initial leaf $\alpha\colon A \longrightarrow B$ and let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ that contains all its objects save A and all its morphisms except those whose source is A. Let $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ be the functor which sends A to B, any morphism f whose source is A to the unique morphism f' for which $f'\alpha = f$ and which is the identity on all other objects and morphisms of $\mathcal{C}$. Then $|F|$ is a homotopy equivalence.*

Proof: Due to analogous reasons as in Lemma 4.4.3, for any object X of $\mathcal{C}'$, (X, id) is a terminal object of the comma category F/X. Hence $|F/X|$ is contractible by Corollary 2.2.8, hence $|F|$ is a homotopy equivalence by Theorem 2.3.1. $\square$

We can apply the technique of 'folding in' leaves on certain sub-categories of $* \backslash \phi$ to reduce their size until they become trivial.

Proposition 4.4.5 *Let $\mathcal{C}$ be a finite, non-empty, connected subcategory of $* \backslash \phi$ such that for all morphisms f, g of $* \backslash \phi$ we have that if $g \circ f$ is contained in $* \backslash \phi$ then so are f and g. Then $|\mathcal{C}|$ is contractible.*

Proof: As $\mathcal{C}$ is finite, it has objects which are minimal and objects which are maximal with respect to the partial order of $\mathcal{C}$. Since for any two objects X and Y of $\mathcal{C}$, the element $X^{-1}Y$ of G has a unique decomposition into generators, at least one of these minimal and maximal objects is the tip of an initial or terminal leaf. We can remove this leaf by Lemma 4.4.4 or Lemma 4.4.3 to obtain a homotopy equivalent yet smaller category, and repeat the procedure until we arrive at the trivial category 1. $\square$

Corollary 4.4.6 *Let M be a free monoid and $\phi\colon M \rightarrowtail G$ its groupification. Then $|*\backslash\phi|$ is contractible.*

Proof: Let $\mathcal{I}$ be the set, ordered by inclusion, of finite, non-empty, connected subcategories $\mathcal{C}$ of $*\backslash\phi$ such that for all morphisms f, g of $*\backslash\phi$ we have that if $g \circ f$ is contained in $\mathcal{C}$ then so are f and g. The classifying space of each member of $\mathcal{I}$ is contractible by Proposition 4.4.5, the inductive limit induced by $\mathcal{I}$ is $\mathcal{C}$ and $\mathcal{I}$ is directed, so we can apply Proposition 2.3.4 to conclude that $|\mathcal{C}|$ is also contractible. $\square$

5 Open problems

The obvious question left unanswered by this thesis is whether it is true for *every* monoid M that its groupification $\phi\colon M \longrightarrow G$ induces a homotopy equivalence between classifying spaces. We only know, from Theorem 4.2.7, that $\pi_1|\phi|$ is an isomorphism. I suspect that $|\phi|$ is indeed a homotopy equivalence, and there are several ways that one might go about proving this.

For example, if we can prove that all higher homotopy groups of $|M|$ are trivial, then $|\phi|$ is a weak homotopy equivalence, and thus by Whitehead's Theorem, a strong homotopy equivalence. We could also try to show, as we have done for the commutative and free cases, that for any monoid M the classifying space of the cocomma category $*\backslash\phi$ is contractible. However, commutativity appears to be crucial in the proofs used for the commutative case, so we need a new approach. One way forward might be as follows.

As classifying spaces are CW complices, it would be sufficient to show that all the homology groups of $|*\backslash\phi|$ are trivial, and then to apply Hurewicz's Theorem. The calculation of the homology groups of the topological realisation of any simplicial set X is aided by the fact that one may let the non-degenerate simplices of X generate the chain groups. To prove that the homology groups of $*\backslash\phi$ are trivial, one might compare its cycles with cycles of the cocomma category $*\backslash \mathrm{id}_M$, which we know to be boundaries as $*\backslash \mathrm{id}_M$ is contractible. This is perhaps easier if we split ϕ into a surjective morphism η and an injective morphism ι (like we did in the commutative case).

For ι, one could also try to generalise the proof used for free monoids in Subsection 4.4. It is then necessary to show that the higher homotopy groups of the classifying space of the groupification map are trivial despite the cycles that inevitably crop up.

On the other hand, a proof that $|*\backslash\phi|$ is not contractible for some monoid M would only provide us with a counterexample if ϕ satisfies the conditions of Theorem 2.3.2 (Quillen's Theorem B). The best place to look for a counterexample might be among monoids whose groupification is trivial, that is those monoids whose classifying space has a trivial fundamental group. In order to prove that the classifying space of such a monoid has non-trivial higher homotopy groups, it would be sufficient to show that any of its homology groups is non-trivial (again per Hurewicz's Theorem).

A task that appears much harder than the one set out in this thesis is determining the homotopy and homology groups of the classifying space of any type of small category. One way to restrict the scope of this question is to try to widen the set of criteria under which the classifying space of a category is contractible. For example, we can combine the fact that categories with initial or terminal objects have contractible classifying spaces with the techniques of Subsection 4.4 to determine a class of categories which can be reduced to a point by a finite number of 'folding' actions, each of which induces a homotopy equivalence. For example, if $\mathcal{C}$ is a category with an object X such that for every object Y of $\mathcal{C}$ there is either a unique morphism from X to Y, or a unique morphism from Y to X, then $|\mathcal{C}|$ is contractible.

Finally, there is a more general classifying space construction for topological categories (which reduces to the present construction in case the topology is discrete). We may ask to what extent the results of this thesis carry over to that more general setting.

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