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A semantical and computational approach to covering-based rough sets

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Abstract

In recent years, our behaviour towards data has changed. As technology evolves, we are more and more able to process large amount of data, often in real time, to the point where data itself becomes the building block of data products. Examples of such data products are music and movie recommendations on websites like Spotify and Netflix, friends recommendations on social media or personalized advertising by local supermarkets.

To this aim, we want to obtain useful knowledge from data. In the early eighties, Zdzisław Pawlak introduced rough set theory in order to obtain information from inconsistent, insufficient and incomplete data, and hence, to obtain knowledge from this data. The basic idea of rough set theory is that it provides a lower and upper approximation of a concept with respect to the indiscernibility between objects based on the data. The lower approximation contains all the elements of the universe certainly belonging to the concept, while the upper approximation contains the elements possibly belonging to the concept. In the original model of Pawlak, an equivalence relation is used to model indiscernibility. Many authors have generalized Pawlak's model by using binary non-equivalence relations, neighborhood operators and coverings.

However, as rough set models are designed to process qualitative or discrete data, it faces limitations when dealing with real-valued data sets. To overcome these limitations, it is interesting to study the hybridization of rough set theory and fuzzy set theory. It was recognized early on that both theories are complementary, rather than competitive. In fuzzy rough set theory, we use a fuzzy relation to formalize the indiscernibility between objects and moreover, the concepts we approximate are fuzzy.

The goal of this work is to provide a systematic, theoretical study of rough set theory and fuzzy rough set theory, unifying existing proposals and developing new ones, with a view to enhance their suitability for machine learning purposes, as well as to investigate desirable properties that may be inherited from the contributing theories.

First, we study rough set theory from a semantical point of view. To this aim, we discuss a new conceptual understanding of rough set models. We recall a semantical approach of Pawlak's rough set model, for which we construct a two-part language. Then, we develop a semantical approach to covering-based rough set models, for which we allow other relations between attribute values than the equality relation to describe the indiscernibility between objects. We illustrate this semantical approach by applying it to dominance-based rough sets. Furthermore, we introduce a semantical approach for decision tables with missing values using Pawlak's rough set model.

Next, we provide a theoretical comparison of different covering-based rough set approximation operators. To this end, we construct a unified framework of dual covering-based approximation operators. We recall existing models and introduce new ones. We study equalities and partial order relations between the different approximation operators, in order to derive insight in the accuracy of the approximation operators. In addition, we study which properties of Pawlak's rough set model are maintained for each of the pairs of dual covering-based approximation operators.

Finally, we discuss different fuzzy rough set models. We study fuzzy coverings and fuzzy neighborhood operators based on fuzzy coverings. Additionally, we introduce a fuzzy rough set model which encapsulates many fuzzy rough set models discussed in literature. Moreover, we examine different fuzzy covering-based rough set models. To end, we discuss noise-tolerant fuzzy rough set models.

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CHAPTER 1

Introduction

In recent years, our behaviour towards data has changed. As technology evolves, we are more and more able to process large amounts of data, often in real time, to the point were data itself becomes the building block of data products [123]. Examples of such data products are music and movie recommendations on websites like Spotify and Netflix, friends recommendations on social media or personalized advertising by local supermarkets. Moreover, we can find it in finance, where data is used for credit ratings, and in government, to optimize transportation.

To this aim, we want to obtain useful knowledge from data. Rough set theory provides us a tool to do this. For example, rough set techniques have been used for tumor classification [23], for image retrieval [104] and for the classification of microarray data [109].

In this chapter, we first provide some background to situate our research. We discuss rough set and fuzzy rough set theory. Next, we state the research objectives of this dissertation. We end with the outline of this work.

1.1 Background

In this section, we present some background to situate our research. We discuss rough set theory and fuzzy rough set theory. We end with some topics in machine learning which use (fuzzy) rough set theory.

1.1.1 Rough set theory

In the early eighties, Zdzisław Pawlak [128] introduced rough set theory in order to obtain information from inconsistent, insufficient and incomplete data, and hence, to obtain knowledge from this data. This problem occurs in many situations where decision making is involved: when diagnosing patients, spam or image classification, learning from examples, pattern recognition, rule-based control, minimizing credit risk, ... [130]. For example, assume the universe U to be a set of patients. We say that the data is

- *inconsistent*, if two patients share exactly the same symptoms, but only one of the patients has a certain disease.
- *insufficient*, if we cannot determine whether a patient has a certain disease given the considered symptoms.
- *incomplete*, if for a patient the result of a symptom is missing or if there are multiple possibilities for the result.

Such data can be represented in an information table. An *information table* is presented by the following tuple:

$$T = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}), \tag{1.1}$$

where U is a finite non-empty set of objects, also called instances, called the universe, At is a finite non-empty set of attributes or features, V_a is the non-empty domain, i.e., the set of possible values, of $a \in At$, and $I_a : U \to V_a$ is an information function for $a \in At$. The table T is called *complete* if all functions I_a are complete, i.e., they each map an object of U to exactly one value in V_a, otherwise T is called incomplete. While an information table has been defined by Pawlak [128] to represent data, decision tables are considered for applications such as classification and regression tasks. The table T is called a *decision table* if the set of attributes At is the union of two disjoint sets C and $\{d\}$, with C the set of conditional attributes and d the decision attribute. Note that sometimes a set of decision attributes D is considered.

Since rough set theory does not require preliminary or additional information about data like probability distributions in probability theory, grades of membership in fuzzy set theory, or mass functions in Dempster-Shafer theory of evidence, it has a major advantage compared to other theories which derive information and knowledge from data [38]. The starting point of the theory is the question whether two objects of the universe are indiscernible from each other based on the available attributes. In Pawlak's original rough set model, this indiscernibility is described by an equivalence relation: two objects are in the same equivalence class if they are indiscernible from each other. Based on this indiscernibility relation, Pawlak constructed the lower and upper approximation operator. Given a subset X of the universe U, the lower approximation of X contains those objects which certainly belong to the set X and the upper approximation of X contains those objects which possibly belong to X. This way, we describe the imprecise information given by the set X by two sets which provides us with more precise information. For example, if X is the set of patients with a particular disease, the lower approximation of X contains those patients which certainly have the disease, while the patients in the upper approximation of X possibly have the disease. Based on this information, we only need to perform an extra examination on the patients in the bounadary region, i.e., the difference between the upper and lower approximation of X. Pawlak's approximation operators have been generalized by considering a general binary relation or neighborhood operator. In this work, we will discuss different generalizations.

In literature, there are three approaches to rough set theory: the axiomatic, constructive and semantical approach. The main difference between them is the function of the approximation operators.

- Axiomatic approach: this approach is used to obtain insight in the algebraic structure of approximation operators. Given a set of axioms, we want to determine which approximation operators this set characterizes. In other words, given a set of preferable properties, which approximation operators will fulfil these properties. In this approach, approximation operators are used as primitive notions.

- Constructive or computational approach: in this approach, approximation operators are derived notions based on the following primitive notions: binary relations, partitions of the universe, coverings of the universe, lattices, Boolean algebras, ... We study for each pair of approximation operators which properties it satisfies. This approach is very suitable for applications, as it allows to construct algorithms based on rough set theory.
- Semantical or conceptual approach: this approach is used to provide insight in the meaning of the concepts of rough set theory. Given a decision table, we want to determine the definable sets related with this table. Based on these definable sets, approximation operators are derived notions in this approach.

In this dissertation, most results can be classified under the constructive approach of rough set theory. Given a certain approximation space, we define pairs of approximation operators, study their properties, and compare them with each other to determine which pairs are suitable for applications. Moreover, we discuss some semantically sound approaches to different rough set models. In this work, we will not consider the axiomatic approach to rough set theory. More details on the axiomatic approach to rough set theory can be found in [98, 181].

For more information regarding rough set theory, we refer to the references of [130]. Note that rough set approximation operators have similarities with operators in modal logic [188], formal concept analysis [184] and topology [134], however, comparison with these theories is outside the scope of this work.

1.1.2 Fuzzy rough set theory

The research of rough set theory has flourished since the early eighties. However, as rough set models are designed to process qualitative or discrete data, it faces limitations when dealing with real-valued data sets [81]. To overcome these limitations, it is interesting to study the hybridization of rough set theory and fuzzy set theory.

Fuzzy set theory is introduced by Zadeh in 1965 [193] to deal with vague concepts and graded indiscernibility. In classical set theory, an object belongs to a concept or it does not belong to the concept. It is either 'yes' or 'no', '1' or '0', 'black' or 'white'. However, in everyday life, not everything is binary. As humans, we think in grey-scales. Concepts such as 'old' or 'tall' cannot be describe with a strict boundary. Most people will agree that a child of height $1m$ is short, and a basketball player of $2m$ is tall, but what about me, a person of height $1m78$? Depending on your own height, you will consider me 'short' or 'tall'. To avoid such subjectiveness, we can describe the property 'tall' by a fuzzy set instead of a classical crisp set.

Formally, a fuzzy set is described as a function from the set of objects to a lattice, often the unit interval $[0,1]$. We will always work with $[0,1]$. Thus, a fuzzy set X on the universe U is a function

$$X: U \to [0,1]: x \mapsto X(x). \tag{1.2}$$

Every object $x \in U$ is associated with its membership degree $X(x) \in [0,1]$. For example, the child of height $1m$ will have degree 0, i.e., 'not tall', the basketball player of height $2m$ will have degree 1, i.e., 'tall' and I will have degree 0.6, i.e., I am 'rather tall'.

Note that there is a fundamental difference between fuzzy set theory and probability theory. For instance, consider the concept 'rain'. When looking up the weather, it will tell you with what chance it will rain today, but it will not tell you the intensity of the rain. With the following sentences, we illustrate this difference:

- The sentence 'there is an 80% chance on rain of degree 0.1' states that it will almost certainly drizzle.
- The sentence 'there is a 10% chance on rain of degree 0.8' states that there is a slight chance on pouring rain.

From early on, it was clear that rough set theory and fuzzy set theory are rather complementary, than competitive. We illustrate this with the following quotations.

> 'A fuzzy set is a class with unsharp boundaries whereas a rough set is a crisp set which is coarsely described.'
>
> Lotfi A. Zadeh in the foreword of [124].

> 'Rough set theory is about the size of the pixels, fuzzy set theory is about the existence of more than two levels of grey.'
>
> Didier Dubois and Henri Prade in the foreword of [129].

Dubois and Prade were two of the first authors to describe a fuzzy rough set model. Such a model uses a fuzzy relation or fuzzy neighborhood operator to describe the indiscernibility relation on the universe U. In other words, two objects x and y are indiscernible to a certain degree. Moreover, the set which contains the imprecise information and which we would like to approximate is a fuzzy set instead of a crisp set.

An illustrative example to apply fuzzy rough set theory to is the prediction of the price of our house given the prices of similar houses. Since we want to determine a real-valued outcome, the approximated sets are fuzzy. Moreover, the indiscernibility between houses is fuzzy: two houses can very similar based on the number of rooms, residential area, type of roof, ... However, if we used a crisp relation to describe the similarity of houses, we would find a very small number of houses similar to ours, hence, it would be very difficult to predict the price of our house.

Fuzzy rough set theory has been applied in many machine learning applications which deal with both vague and incomplete information. In the next section, we discuss some of these machine learning techniques, to illustrate the motivation behind the multiple developments of different fuzzy rough set models.

1.1.3 Applications in machine learning

In [162], Vluymans et al. present a survey of applications of fuzzy rough set theory in machine learning. We will highlight some machine learning applications, to situate the research domain. As we will not discuss applications in this dissertation, we will not provide details for which we refer to [162].

The goal of machine learning applications is to obtain knowledge from data. To this aim, algorithms which can learn from data and predict based on known properties are studied and constructed. This way, machine learning slightly differs from data mining, which focuses more on discovering patterns in data.

Fuzzy rough set theory is used in a wide variety of machine learning applications. Here we will focus on some supervised learning algorithms. Supervised learning implies that a set of labeled elements is available in the training phase of the method [162], i.e., there is a teacher at hand for the method to learn from. A supervised learning algorithm constructs a model to predict the outcome of new objects based a set of labeled objects, i.e., objects of which we know the outcome. To go back to the pricing of our house, we want to predict the price based on a set of houses of which we know the price.

If the outcome of the prediction is categorical, the learning task is considered a classification task, otherwise, we consider it a regression task. We will discuss both tasks briefly. Moreover, the data we consider often needs to be preprocessed, in order to fasten the learning time or to increase the prediction ability. Examples of such preprocessing step are feature and instance selection, which we discuss first.

Feature selection

Feature or *attribute selection* is a preprocessing step which removes conditional attributes from the decision table, in order to decrease the dimensionality of the problem without losing the semantical meaning of the data. In addition, we would like our reduced set of conditional attributes to preserve the ability to discern the objects of the universe which are discernible based on the original data. The goal is to obtain a set of conditional attributes which is minimal for this property. Such a minimal set is called a *reduct*. We say that the conditional attributes in such a reduct are joint sufficient and individually necessary, i.e., together they preserve the discernibility ability of the original set of conditional attributes and moreover, none of the attributes can be removed from this set, i.e., they are all necessary. When a set of conditional attributes which preserves the discernibility ability is not minimal, it is called a *superreduct*.

In a classification problem, it is usually sufficient to preserve the ability to discern between decision classes and not necessarily between individual elements. In this case, we call such a minimal set a *decision reduct* or *relative reduct*.

Additionally, a general framework for fuzzy rough attribute selection was presented in [18]. Let $\mathscr{M}$ be a monotone measure to determine the degree of ability to preserve the discernibility between decision classes such that $\mathscr{M}(C) = 1$. Let $A \subseteq C$, then A is a *fuzzy $\mathscr{M}$-decision superreduct to degree $\alpha \in (0,1]$* if $\mathscr{M}(A) \geq \alpha$. If A is minimal for this property for α, i.e., $\forall A' \subseteq A$: $\mathscr{M}(A') < \alpha$, the set A is called a *fuzzy $\mathscr{M}$-decision reduct to degree α*.

To determine which attributes are dispensable or redundant, we can consider different techniques. If we want to determine all reducts, we could make use of a discernibility matrix. If finding one reduct is sufficient, we could use some heuristic algorithms.

A *discernibility matrix M* is a matrix of size $n \times n$, with $n = |U|$. Every entry M_{ij} of the matrix contains those conditional attributes which discern the objects $x_i, x_j \in U$. Based on M, we can describe the *discerniblity function f* given by

$$f = \bigwedge \left\{ \bigvee \{a \in C \mid a \in M_{ij}\} \mid M_{ij} \neq \emptyset \right\},$$

i.e., we take the conjunction of all disjunctions of conditional attributes in a non-empty entry of the matrix. Such discernibility function is defined as a conjunctive normal form. If we transform f into its disjunctive normal form, every disjunction of f represents a decision reduct. Note that this transformation is an NP-hard problem.

In literature, the concepts of discernibility matrix and discernibility function are extended to fuzzy rough set theory. Examples of algorithms which use a fuzzy discernibility matrix to determine all fuzzy decision reducts can be found in [11, 12, 14, 66, 195].

On the other hand, if we only want to determine one decision reduct, it is sufficient to apply a heuristic algorithm. For this, the dependency degree [18] is often used as a measure to express the portion of the universe which can be discerned when reducing the set of conditional attributes. For example, the QuickReduct algorithm [15] uses the dependency degree as search and stopping criterium: starting from the empty set, the conditional attribute which provides the highest dependency degree is added to the set of attributes. The result of this algorithm is a superreduct. Examples which uses a fuzzy variant of QuickReduct of an alternative heuristic algorithm can be found in [5, 16, 17, 37, 39, 82, 83, 108, 133, 167].

Instance selection

Instance selection is a preprocessing step similar to feature selection, however, the size of the universe is reduced instead of the size of the attribute set. By removing instances or objects from the data, it can remove noisy data and thus, improve the performance of the learning algorithm. Other advantages are the reduced storage requirement and reduced running times. This preprocessing step is also called prototype selection, training set selection and sample selection.

The first instance selection method using fuzzy rough set theory was developed in [78]. The authors considered three different decision criteria to decide whether an object is removed from the dataset. A more complex instance selection method was proposed in [159] and later optimized in [160], for which the authors used ordered weighted average based fuzzy rough sets [19].

Classification

In classification tasks, the goal is to predict a qualitative outcome. Given a training set where each object is labeled to a decision class, we want to determine for a new object to which decision class it belongs, by comparing it based on the conditional attributes to the objects in the training set.

A possible classification algorithm is rule induction [60]. Based on the data at hand, we want to construct ‘if-then’ rules. Applications which discuss fuzzy rule induction can be found in [50, 67, 80, 102, 157, 167, 195].

Another classification algorithm is nearest neighbor classification. Given $k \in \mathbb{N}$, the k nearest neighbor classifier [22] assigns a class label to a new object by looking at the k nearest training instances. In [6, 85, 136, 139, 145, 161], we can find some methods using fuzzy k nearest neighbor algorithms.

Regression

In regression tasks, we want to estimate a real-valued outcome based on the conditional attributes values of the object. Since less attention has been directed to fuzzy rough set theory based regression, research on this topic is minor. Applications can be found in [1, 79].

1.2 Research objectives

The goal of this work is to provide a systematic, theoretical study of rough set theory and fuzzy rough set theory, unifying existing proposals and developing new ones, with a view to enhance their suitability for machine learning purposes, as well as to investigate desirable properties that may be inherited from the contributing theories. Based on the theoretical study done in this dissertation, researchers are able to choose between different models based on their needs. In order to do so, we define the following research objectives:

- In a wide range of generalizations of Pawlak's rough set model, which generalizations are meaningful from a semantical point of view? In other words, we will study which approximation operators provide approximations which can be interpreted by means of the original data.
- In a wide range of (fuzzy) generalizations of Pawlak's rough set model, which generalizations are meaningful from a constructive point of view? In other words, we will study which approximation operators maintain theoretical properties.
- In a wide range of (fuzzy) generalizations of Pawlak's rough set model, which generalizations are meaningful from a practical point of view? In other words, we will study which approximation operators yield high accuracy.

1.3 Outline of the dissertation

The outline of this work is as follows. In Chapter 2, we introduce some preliminary notions of rough set theory. First, we discuss the original rough set model introduced by Pawlak [128]. The model of Pawlak has three equivalent definitions which can all be generalized in different ways. Such generalizations can often be classified as covering-based rough set models. In Section 2.2, we recall different covering-based rough set models defined in literature. Furthermore, we discuss the variable precision rough set (VPRS) model of Ziarko [203]. This model allows for more flexibility towards noisy data compared to Pawlak's rough set model.

In Chapter 3, we study rough set theory from a semantical point of view. To this aim, we discuss a new conceptual understanding of rough set models. We recall a semantical approach of Pawlak's rough set model, for which we construct a two-part language. The formulas related with this language can be interpreted as the intension of a concept [186]. The extensions of the concepts related with the formulas provide the set of definable sets, which is algebraically described by a Boolean algebra. Then, we develop a semantical approach to covering-based rough set models. Since we allow other relations between attribute values than the equality relation, the set of definable sets will now longer be given by a Boolean algebra, but by a join-semilattice. We illustrate this semantical approach by applying it to dominance-based rough sets. Furthermore, we introduce a semantical approach for decision tables with missing values using Pawlak's rough set model. The results obtained in Sections 3.3 and 3.5 motivate the research on covering-based rough set models, as it allows rule induction for ordered and incomplete decision tables.

In Chapter 4, we aim to provide a theoretical comparison of different covering-based approximation operators. To this end, we construct a unified framework of dual covering-based approximation operators. By studying equalities between different approximation operators, we reduce the 69 pairs of dual approximation operators we consider to 36 pairs of covering-based rough set approximation operators, of which 13 pairs are introduced by us. Furthermore, we study partial order relations between these 36 pairs to provide more insight in the accuracy of the approximation operators. In addition, we study which properties of Pawlak's rough set model are maintained for each of the 36 pairs.

In Chapter 5, we recall some preliminary notions of fuzzy set theory. First, we discuss fuzzy logical connectives and fuzzy set theory introduced by Zadeh [193]. Moreover, we recall some aggregation operators and we discuss the technique of representation by levels introduced by Sánchez et al. [144].

In Chapter 6, we discuss fuzzy neighborhood operators from the perspective of fuzzy rough set theory. We introduce the notion of a fuzzy covering, the fuzzy neighborhood system of an object and the fuzzy minimal and maximal description of an object given a fuzzy covering. In addition, we extend four crisp neighborhood operators and six crisp coverings studied in [189] to the fuzzy setting. We study which results for crisp neighborhood operators are maintained. Moreover, we combine four fuzzy neighborhood operators and six fuzzy coverings, resulting in 24 combinations of fuzzy neighborhood operators. We study partial order relations between these 24 fuzzy neighborhood operators. In addition, we discuss a fuzzy neighborhood operator introduced in [107].

In Chapter 7, we introduce the implicator-conjunctor-based (IC) fuzzy rough set model. First, we present a historical overview on the research of fuzzy rough set theory. Next, we introduce this general fuzzy rough set model which encapsulates many fuzzy rough set models discussed in literature and we study which properties of Pawlak's rough set model are maintained for the IC model. Note that each fuzzy neighborhood operator studied in Chapter 6 may be used to describe the indiscernibility between objects of the universe, hence, it may be used to define the IC model.

In Chapter 8, we examine different fuzzy covering-based rough set models. We recall four existing models and introduce three new ones. For each of the seven models, we discuss which properties of Pawlak's rough set model are maintained. Furthermore, we discuss partial order relations between these seven models and IC models using different fuzzy neighborhood operators based on a fuzzy covering to study accuracy of the approximation operators.

In Chapter 9, we discuss different noise-tolerant fuzzy rough set models. A drawback of the IC model is the use of the infimum and supremum operator, as both operators are very sensitive to noise in data. Here, we recall seven noise-tolerant fuzzy rough set models: four models are frequency-based, one adjusts the set which is approximated and two use different aggregation operators than the infimum and supremum operator. For each model, we generalize, correct or simplify its definition and study which properties of Pawlak's rough set model are maintained. Similarly as the VPRS model, allowing more flexibility towards noise typically involves sacrificing some desirable properties. Furthermore, we analyze the robustness of the noise-tolerant fuzzy rough set models and the IC model with respect to attribute and class noise.

Finally, in Chapter 10, we state the most important conclusions of this work. In addition, we discuss future research challenges.

Many results presented in this dissertation have been published or have been submitted for publication to peer reviewed international journals and to the proceedings of international conferences. Specifically, the semantical approach to Pawlak's rough set model and covering-based rough set models has been presented in [30], and the application to dominance-based rough sets was discussed in [29]. The publication of the results concerning the semantical approach for decision tables with missing values is in preparation. The computational approach of rough sets was first studied in [31] and later extended in [26]. Currently, we are preparing the publication of the results concerning the frameworks of Zhao and Samanta and Chakraborty. The study on fuzzy neighborhood operators was published in [27]. We introduced the implicator-conjunctor-based fuzzy rough set model in the journal article [35], in which we also discussed noise-tolerant fuzzy rough set models, and in the conference proceedings [33, 34]. Some noise-tolerant models were adapted in [32]. Finally, fuzzy covering-based rough set models were studied in [25, 28].

CHAPTER 2

Preliminaries

In this chapter, we introduce some preliminary notions of rough set theory. In Section 2.1, we recall the original rough set model of Pawlak. Next, in Section 2.2, we discuss different covering-based rough set models defined in literature. Finally, we recall the variable precision rough set model of Ziarko in Section 2.3.

2.1 Pawlak's rough set model

In the early eighties, Zdzisław Pawlak [128] introduced rough set theory in order to obtain information from inconsistent, insufficient and incomplete data. This problem occurs in many situations where decision making is involved: when diagnosing patients, spam or image classification, learning from examples, pattern recognition, rule-based control, minimizing credit risk, ... [130]. For example, assume the universe U to be a set of patients. We say that the data is

- *inconsistent*, if two patients share exactly the same symptoms, but only one of the patients has a certain disease.

- *insufficient*, if we cannot determine whether a patient has a certain disease given the considered symptoms.
- *incomplete*, if for a patient the result of a symptom is missing or if there are multiple possibilities for the result.

When two patients share the same symptoms, we say that these patients are *indiscernible* from each other, i.e., we cannot distinguish them based on the list of symptoms. More formally, Pawlak used an equivalence relation to describe the indiscernibility between objects, also called instances, of the universe. Recall that an equivalence relation $E \subseteq U \times U$ is

- reflexive, i.e., $\forall x \in U \colon (x,x) \in E$,
- symmetric, i.e., $\forall x, y \in U \colon (x,y) \in E \Rightarrow (y,x) \in E$,
- transitive, i.e., $\forall x, y, z \in U \colon (x,y) \in E$ and $(y,z) \in E \Rightarrow (x,z) \in E$.

The equivalence class of an instance $x \in U$ by E describes all the objects which cannot be discerned from x and is denoted by

$$[x]_E = \{y \in U \mid (x,y) \in E\}.$$

Given such an equivalence relation E, we call the tuple (U,E) a *Pawlak approximation space*. Note that the equivalence classes are also called the *basic granules* of U.

Furthermore, let $X \subseteq U$ denote a concept in the Pawlak approximation space (U,E), e.g., X is the set of all patients with a particular disease. It is possible that we cannot fully describe the concept X by the equivalence classes given in the partition U/E. Therefore, Pawlak introduced the *lower* and *upper approximation operator* with respect to the indiscernibility relation E.

Definition 2.1.1. Let (U,E) be a Pawlak approximation space, then the lower approximation operator $\underline{\text{apr}}_E \colon \mathscr{P}(U) \to \mathscr{P}(U)$ is defined by

$$\forall X \subseteq U \colon \underline{\text{apr}}_E(X) = \{x \in U \mid [x]_E \subseteq X\} \tag{2.1}$$

and the upper approximation operator $\overline{\text{apr}}_E \colon \mathscr{P}(U) \to \mathscr{P}(U)$ is defined by

$$\forall X \subseteq U \colon \overline{\text{apr}}_E(X) = \{x \in U \mid [x]_E \cap X \neq \emptyset\}, \tag{2.2}$$

where $\mathscr{P}(U)$ represents the collection of subsets of U.

The lower approximation $\underline{\text{apr}}_E(X)$ of a concept X contains those objects which certainly belong to X and the upper approximation $\overline{\text{apr}}_E(X)$ of a concept X contains those objects which possibly belong to X. Going back to our example where X is the set of patients with a particular disease, we see that the patients in $\underline{\text{apr}}_E(X)$ certainly have the disease, while the patients in $\overline{\text{apr}}_E(X)$ possibly have the disease.

Besides the lower and upper approximation of a concept X, we can also describe X by three pair-wise disjoint regions: the positive, the negative and the boundary region [129]. Let $X \subseteq U$, then the positive region, the negative region and the boundary region of X in (U, E) are defined by

$$\text{POS}_E(X) = \underline{\text{apr}}_E(X), \tag{2.3}$$
$$\text{NEG}_E(X) = \underline{\text{apr}}_E(X^c), \tag{2.4}$$
$$\text{BND}_E(X) = \overline{\text{apr}}_E(X) \setminus \underline{\text{apr}}_E(X), \tag{2.5}$$

where X^c represents the set-theoretic complement of X. The positive region of X is the same as the lower approximation of X, it describes all objects certainly belonging to the concept, while the negative region describes the objects which certainly not belong to the concept (in our example it describes all the patients which certainly do not have the disease). The boundary region contains those elements of the universe of which it is uncertain whether they belong to the concept X or not.

Note that the lower and upper approximation operators defined in Definition 2.1.1 are referred to as the two-way approximation operators and the three regions are referred to as the three-way approximation operators. Furthermore, we are able to obtain the two-way approximation operators from the three-way approximation operators: for $X \subseteq U$, we have that

$$\begin{aligned} \underline{\text{apr}}_E(X) &= \text{POS}_E(X), \\ \overline{\text{apr}}_E(X) &= \text{POS}_E(X) \cup \text{BND}_E(X). \end{aligned}$$

The interpretation of the two- and three-way approximation operators with respect to E is illustrated in Figure 2.1.

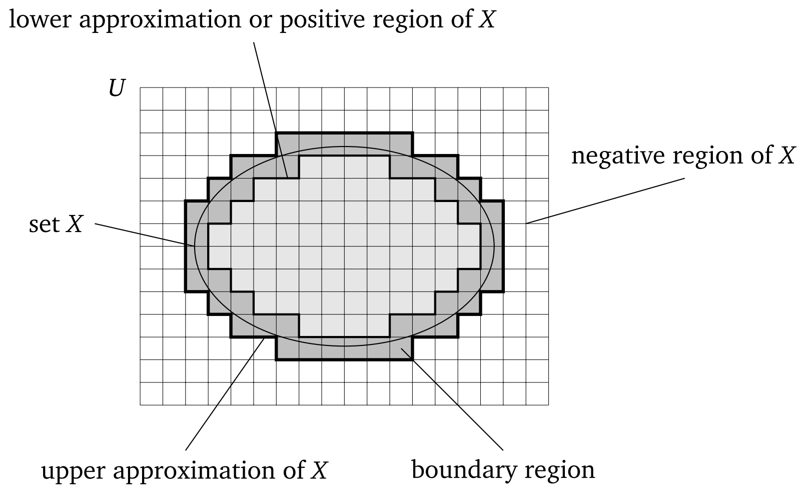


Figure 2.1: Two- and three-way approximations of a concept X in (U, E)

The two-way approximation operators defined in Eqs. (2.1) and (2.2) have equivalent characterizations [189]. While in Definition 2.1.1 the *element-based* definition of Pawlak's rough set model is given, we can also consider the *granule-based* definition

$$\forall X \subseteq U\colon \underline{\mathrm{apr}}_E(X) = \bigcup\{[x]_E \in U/E \mid [x]_E \subseteq X\}, \tag{2.6}$$

$$\forall X \subseteq U\colon \overline{\mathrm{apr}}_E(X) = \bigcup\{[x]_E \in U/E \mid [x]_E \cap X \neq \emptyset\} \tag{2.7}$$

or the *subsystem-based* definition

$$\forall X \subseteq U\colon \underline{\mathrm{apr}}_E(X) = \bigcup\{Y \in \mathscr{B}(U/E) \mid Y \subseteq X\}, \tag{2.8}$$

$$\forall X \subseteq U\colon \overline{\mathrm{apr}}_E(X) = \bigcap\{Y \in \mathscr{B}(U/E) \mid X \subseteq Y\}. \tag{2.9}$$

In this definition, $\mathscr{B}(U/E) \subseteq \mathscr{P}(U)$ is obtained from the partition U/E by adding the empty set and closing it under set union. It is a σ-algebra of subsets of U and it is a sub-Boolean algebra of the Boolean algebra $(\mathscr{P}(U), {}^c, \cap, \cup)$, where $\cap$ and $\cup$ represent the set intersection and set union [128].

To end this subsection, we discuss the properties of Pawlak's rough set approximation operators in Table 2.1, where X and Y denote arbitrary subsets of U. Note that with each property we associate an abbreviation.

The properties (D), (INC), (SM), (IU), (ID), (LU) and (UE) were already stated in [128].

The idea of the adjointness property (A) comes from the concept of a Galois connection [76, 158]: let U_1 and U_2 be two universes and $f : U_1 \to U_2$ and $g : U_2 \to U_1$ two mappings between these universes. We call (f, g) a *Galois connection* in (U_1, U_2) if for all $x \in U_1$ and $y \in U_2$ it holds that $f(x) \leq_2 y \Leftrightarrow x \leq_1 g(y)$, where $\leq_i$ is an order relation in U_i. It is clear that if f and g are the upper and lower approximation operator respectively, then they form a Galois connection in $(\mathscr{P}(U), \mathscr{P}(U))$ with $\leq_1 = \leq_2 = \subseteq$. As a consequence of this property, if for $X \subseteq U$ $\underline{\mathrm{apr}}_E(X) = X$ holds, then necessarily also $\overline{\mathrm{apr}}_E(X) = X$, and vice versa.

The (RM)-property is relevant in many applications of rough sets. In particular, in the context of attribute selection in information systems [146,154], the relation E represents the indiscernibility between objects based on their attribute values. When attributes are omitted from the information system, the granularity imposed by the equivalence relation becomes coarser and it is then desirable that the lower approximation does not increase, and the upper approximation does not decrease.

2.2 Covering-based rough set models

In this section, we discuss different covering-based rough set models defined in literature. We present the frameworks of Yao and Yao [189], Yang and Li [177], Zhao [197] and Samanta and Chakraborty [142, 143]. Furthermore, we discuss the concepts of singleton, subset and concept approximation operators.

2.2.1 Framework of Yao and Yao

We start by discussing the covering-based approximation operators studied by Yao and Yao [189]. By weakening the condition of an equivalence relation, many generalizations of Pawlak's rough set model can be defined. An important generalization can be obtained by replacing the partition U/E with a covering of U.

Table 2.1: Properties of the lower and upper approximation operators for Pawlak approximation spaces (U, E), (U, E_1) and (U, E_2)

Property	Abbreviation	Definition
Duality	(D)	$\overline{\mathrm{apr}}_E(X) = (\underline{\mathrm{apr}}_E(X^{\mathrm{c}}))^{\mathrm{c}}$
Inclusion	(INC)	$\underline{\mathrm{apr}}_E(X) \subseteq X$ and $X \subseteq \overline{\mathrm{apr}}_E(X)$
Set monotonicity	(SM)	$X \subseteq Y \Rightarrow \begin{cases} \underline{\mathrm{apr}}_E(X) \subseteq \underline{\mathrm{apr}}_E(Y) \\ \overline{\mathrm{apr}}_E(X) \subseteq \overline{\mathrm{apr}}_E(Y) \end{cases}$
Intersection and union	(IU)	$\underline{\mathrm{apr}}_E(X \cap Y) = \underline{\mathrm{apr}}_E(X) \cap \underline{\mathrm{apr}}_E(Y)$ $\overline{\mathrm{apr}}_E(X \cup Y) = \overline{\mathrm{apr}}_E(X) \cup \overline{\mathrm{apr}}_E(Y)$
Idempotence	(ID)	$\underline{\mathrm{apr}}_E(\underline{\mathrm{apr}}_E(X)) \supseteq \underline{\mathrm{apr}}_E(X)$ $\overline{\mathrm{apr}}_E(\overline{\mathrm{apr}}_E(X)) \subseteq \overline{\mathrm{apr}}_E(X)$
Interaction lower and upper	(LU)	$\overline{\mathrm{apr}}_E(\underline{\mathrm{apr}}_E(X)) \subseteq \underline{\mathrm{apr}}_E(X)$ $\underline{\mathrm{apr}}_E(\overline{\mathrm{apr}}_E(X)) \supseteq \overline{\mathrm{apr}}_E(X)$
Universe and empty set	(UE)	$\underline{\mathrm{apr}}_E(U) = U$ and $\overline{\mathrm{apr}}_E(U) = U$ $\overline{\mathrm{apr}}_E(\emptyset) = \emptyset$ and $\overline{\mathrm{apr}}_E(\emptyset) = \emptyset$
Adjointness	(A)	$\overline{\mathrm{apr}}_E(X) \subseteq Y \Leftrightarrow X \subseteq \underline{\mathrm{apr}}_E(Y)$
Relation monotonicity	(RM)	$E_1 \subseteq E_2 \Rightarrow \begin{cases} \underline{\mathrm{apr}}_{E_2}(X) \subseteq \underline{\mathrm{apr}}_{E_1}(X) \\ \overline{\mathrm{apr}}_{E_1}(X) \subseteq \overline{\mathrm{apr}}_{E_2}(X) \end{cases}$

Definition 2.2.1. [201] Let $\mathbb{C} = \{K_i \subseteq U \mid K_i \neq \emptyset, i \in I\}$ be a family of non-empty subsets of U, with I a set of indices. The set $\mathbb{C}$ is called a *covering* of U if $\bigcup_{i\in I} K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a *covering approximation space*.

It is clear that a partition generated by an equivalence relation is a special case of a covering of U. Every set $K \in \mathbb{C}$ is called a *patch*. In a covering approximation space, equivalence classes can be generalized to neighborhoods.

Definition 2.2.2. [189] A neighborhood operator in a covering approximation space $(U, \mathbb{C})$ is a mapping $\mathrm{N}\colon U \to \mathscr{P}(U)$ which associates with every object $x \in U$ a neighborhood $\mathrm{N}(x) \subseteq U$.

In general, it is assumed that a neighborhood operator N is reflexive, i.e., $\forall x \in U\colon x \in \mathrm{N}(x)$, in order to fulfil the intuitive idea of a *neighborhood*. Furthermore, a neighborhood operator N can be symmetric, i.e.,

$$\forall x, y \in U\colon x \in \mathrm{N}(y) \Rightarrow y \in \mathrm{N}(x),$$

and it can be transitive, i.e.,

$$\forall x, y \in U\colon x \in \mathrm{N}(y) \Rightarrow \mathrm{N}(x) \subseteq \mathrm{N}(y).$$

Moreover, given a neighborhood operator N, we can define the *inverse neighborhood operator* N^{-1} by $\forall x, y \in U\colon y \in \mathrm{N}^{-1}(x) \Leftrightarrow x \in \mathrm{N}(y)$.

The neighborhood of an object $x \in U$ can be regarded as a generalization of the equivalence class $[x]_E$. Therefore, each neighborhood operator N defines an ordered pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ of element-based approximation operators defined by, for $X \subseteq U$,

$$\underline{\mathrm{apr}}_{\mathrm{N}}(X) = \{x \in U \mid \mathrm{N}(x) \subseteq X\}, \quad (2.10)$$
$$\overline{\mathrm{apr}}_{\mathrm{N}}(X) = \{x \in U \mid \mathrm{N}(x) \cap X \neq \emptyset\}. \quad (2.11)$$

Yao and Yao [189] described some neighborhood operators based on a covering $\mathbb{C}$. For this purpose, they defined the *neighborhood system* $\mathscr{C}(\mathbb{C}, x)$ of an element $x \in U$ given the covering $\mathbb{C}$ as follows:

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} \mid x \in K\}. \quad (2.12)$$

In a neighborhood system $\mathscr{C}(\mathbb{C},x)$, the minimal and maximal sets which contain an element $x \in U$ are particularly important. The set

$$\mathrm{md}(\mathbb{C},x) = \{K \in \mathscr{C}(\mathbb{C},x) \mid (\forall S \in \mathscr{C}(\mathbb{C},x))(S \subseteq K \Rightarrow K = S)\} \tag{2.13}$$

is called the *minimal description* of x [8]. On the other hand, the set

$$\mathrm{MD}(\mathbb{C},x) = \{K \in \mathscr{C}(\mathbb{C},x) \mid (\forall S \in \mathscr{C}(\mathbb{C},x))(K \subseteq S \Rightarrow K = S)\} \tag{2.14}$$

is called the *maximal description* of x [202]. The sets $\mathrm{md}(\mathbb{C},x)$ and $\mathrm{MD}(\mathbb{C},x)$ are also called the minimal-description and maximal-description neighborhood systems of x [189]. The importance of the minimal and maximal description of x is demonstrated by the following proposition:

Proposition 2.2.3. [189] Let $(U,\mathbb{C})$ be a covering approximation space, $x \in U$ and $K \in \mathscr{C}(\mathbb{C},x)$.

(a) If any descending chain in $\mathbb{C}$ is closed under the infimum, i.e., if for any set $\{K_i \mid i \in I\}$ with $K_{i+1} \subseteq K_i$ it holds that $\inf_{i\in I} K_i = \bigcap_{i\in I} K_i \in \mathbb{C}$, then there exists a set $K_1 \in \mathrm{md}(\mathbb{C},x)$ with $K_1 \subseteq K$. Moreover, it holds that

$$\bigcap\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C},x)\} = \bigcap\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C},x)\}.$$

(a) If any ascending chain in $\mathbb{C}$ is closed under the supremum, i.e., if for any set $\{K_i \mid i \in I\}$ with $K_i \subseteq K_{i+1}$ it holds that $\sup_{i\in I} K_i = \bigcup_{i\in I} K_i \in \mathbb{C}$, then there exists a set $K_2 \in \mathrm{MD}(\mathbb{C},x)$ with $K \subseteq K_2$. Moreover, it holds that

$$\bigcup\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C},x)\} = \bigcup\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C},x)\}.$$

Note that the conditions in Proposition 2.2.3 are necessary, as illustrated in the next example.

Example 2.2.4. Let $U = (-1,1)$, $x = 0 \in U$ and assume

$$\mathscr{C}(\mathbb{C},x) = \{(-\frac{1}{n},\frac{1}{n}) \mid n \in \mathbb{N} \setminus \{0\}\}.$$

As for all $n \in \mathbb{N}\setminus\{0\}$ it holds that $(-\frac{1}{n+1},\frac{1}{n+1}) \subsetneq (-\frac{1}{n},\frac{1}{n})$, we obtain that $\mathrm{md}(\mathbb{C},x) = \emptyset$. Moreover, it holds that

$$(-1,1) = U = \bigcap\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C},x)\} \neq \bigcap\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C},x)\} = \{0\}.$$

Note that Proposition 2.2.3 is satisfied when the covering $\mathbb{C}$ is finite. We will always assume that the conditions on $\mathbb{C}$ are satisfied.

Given the three neighborhood systems of $x \in U$, Yao and Yao [189] constructed the following four neighborhood operators based on the covering $\mathbb{C}$:

- $N_1^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C}, x)\} = \bigcap\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C}, x)\}$,
- $N_2^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} \mid K \in \mathrm{md}(\mathbb{C}, x)\}$,
- $N_3^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\}$,
- $N_4^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\} = \bigcup\{K \in \mathbb{C} \mid K \in \mathscr{C}(\mathbb{C}, x)\}$.

Therefore, for each $N_i^{\mathbb{C}}$ with $i = 1, 2, 3, 4$, we have a pair of dual approximation operators $\left(\underline{\mathrm{apr}}_{N_i^{\mathbb{C}}}, \overline{\mathrm{apr}}_{N_i^{\mathbb{C}}}\right)$ defined in Eqs. (2.10) and (2.11).

A large portion of element-based approximation operators studied in literature uses a binary relation R on U to describe the neighborhood of an element x. More specifically, given a relation $R \subseteq U \times U$ and an object $x \in U$, then the *R-foreset* $R^p(x)$ of x is given by all the predecessors of x by R:

$$R^p(x) = \{y \in U \mid (y, x) \in R\} \tag{2.15}$$

and the *R-afterset* $R^s(x)$ of x is given by all the successors of x by R:

$$R^s(x) = \{y \in U \mid (x, y) \in R\}. \tag{2.16}$$

Often the considered relation R is assumed to be reflexive. If the relation R is reflexive and symmetric, it is called a *tolerance relation* [147] and if R is reflexive and transitive, then R is called a *pre-order* or *dominance relation* [53]. Note that a binary relation R can also satisfy the following properties:

- R is serial if and only if $\forall x \in U, \exists y \in U : (x, y) \in R$.
- R is inverse serial if and only if $\forall x \in U, \exists y \in U : (y, x) \in R$.
- R is Euclidean if and only if $\forall x, y, z \in U : (x, y) \in R, (x, z) \in R \Rightarrow (y, z) \in R$.

Note that a symmetric relation R is Euclidean if and only if it is transitive. Moreover, if R is reflexive and Euclidean, it is also symmetric and thus an equivalence relation.

It is clear that when R is an equivalence relation, the element-based approximation operators defined in Eqs. (2.10) and (2.11) with $\mathrm{N}(x) = [x]_R$ for $x \in U$ coincide with Pawlak approximation operators. When a binary relation R is used to describe the discernibility between the objects of the universe, we consider a *relation approximation space* (U, R) instead of a covering approximation space $(U, \mathbb{C})$. Moreover, when for example the foresets are considered, the approximation operators are then denoted by, for $X \subseteq U$,

$$\underline{\mathrm{apr}}_R(X) = \{x \in U \mid \forall y \in U\colon (y,x) \in R \Rightarrow y \in X\}, \tag{2.17}$$

$$\overline{\mathrm{apr}}_R(X) = \{x \in U \mid \exists y \in U\colon (y,x) \in R \wedge y \in X\}. \tag{2.18}$$

Besides generalizations of the element-based definition of Pawlak's rough set model, the granule-based representations can be generalized by considering a covering $\mathbb{C}$ instead of a partition U/E. However, although $(\underline{\mathrm{apr}}_E, \overline{\mathrm{apr}}_E)$ are dual approximation operators, this property is no longer satisfied for the generalizations. Therefore, given a covering $\mathbb{C}$, we consider two pairs of dual approximation operators $\left(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}}\right)$ and $\left(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}}\right)$ which are defined, for $X \subseteq U$, by

$$\begin{aligned}
\underline{\mathrm{apr}}'_{\mathbb{C}}(X) &= \bigcup\{K \in \mathbb{C} \mid K \subseteq X\} && (2.19)\\
&= \{x \in U \mid (\exists K \in \mathbb{C})(x \in K \wedge K \subseteq X)\}\\
\overline{\mathrm{apr}}'_{\mathbb{C}}(X) &= (\underline{\mathrm{apr}}'_{\mathbb{C}}(X^{\mathrm{c}}))^{\mathrm{c}} && (2.20)\\
&= \{x \in U \mid (\forall K \in \mathbb{C})(x \in K \Rightarrow K \cap X \neq \emptyset)\}\\
\underline{\mathrm{apr}}''_{\mathbb{C}}(X) &= (\overline{\mathrm{apr}}''_{\mathbb{C}}(X^{\mathrm{c}}))^{\mathrm{c}} && (2.21)\\
&= \{x \in U \mid (\forall K \in \mathbb{C})(x \in K \Rightarrow K \subseteq X)\}\\
\overline{\mathrm{apr}}''_{\mathbb{C}}(X) &= \bigcup\{K \in \mathbb{C} \mid K \cap X \neq \emptyset\} && (2.22)\\
&= \{x \in U \mid (\exists K \in \mathbb{C})(x \in K \wedge K \cap X \neq \emptyset)\}
\end{aligned}$$

The pair $\left(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}}\right)$ is called the tight pair of covering-based approximation operators, while $\left(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}}\right)$ is called the loose pair of covering-based approximation operators [21]. This is because of the following property:

$$\forall X \subseteq U\colon \underline{\mathrm{apr}}''_{\mathbb{C}}(X) \subseteq \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \subseteq X \subseteq \overline{\mathrm{apr}}'_{\mathbb{C}}(X) \subseteq \overline{\mathrm{apr}}''_{\mathbb{C}}(X).$$

Besides neighborhood operators based on a covering $\mathbb{C}$, Yao and Yao [189] also considered six coverings derived from an initial covering $\mathbb{C}$:

- $\mathbb{C}_1 = \bigcup\{\mathrm{md}(\mathbb{C},x) \mid x \in U\}$,
- $\mathbb{C}_2 = \bigcup\{\mathrm{MD}(\mathbb{C},x) \mid x \in U\}$,
- $\mathbb{C}_3 = \left\{\bigcap \mathrm{md}(\mathbb{C},x) \mid x \in U\right\} = \left\{\bigcap \mathscr{C}(\mathbb{C},x) \mid x \in U\right\}$,
- $\mathbb{C}_4 = \left\{\bigcup \mathrm{MD}(\mathbb{C},x) \mid x \in U\right\} = \left\{\bigcup \mathscr{C}(\mathbb{C},x) \mid x \in U\right\}$,
- $\mathbb{C}_\cap = \mathbb{C} \setminus \left\{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})\left(K = \bigcap \mathbb{C}'\right)\right\}$,
- $\mathbb{C}_\cup = \mathbb{C} \setminus \left\{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})\left(K = \bigcup \mathbb{C}'\right)\right\}$.

The idea behind the first two coverings is similar to the rationale for $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. Given the extreme neighborhood systems $\mathrm{md}(\mathbb{C},x)$ and $\mathrm{MD}(\mathbb{C},x)$ for $x \in U$, the union of these systems leads to new coverings. Note that this is not the case when taking the intersection. Coverings $\mathbb{C}_3$ and $\mathbb{C}_4$ are directly related with $N_1^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. The covering $\mathbb{C}_3$ is called the *induced covering of* $\mathbb{C}$ and is also denoted by $\mathrm{Cov}(\mathbb{C})$ [10]. Covering $\mathbb{C}_\cap$ is called the *intersection reduct* and $\mathbb{C}_\cup$ the *union reduct*. These reducts eliminate intersection reducible elements, resp. union reducible elements, from the covering, respectively. An intersection reducible element of a covering $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that there exists a subcovering $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ for which $K = \bigcap \mathbb{C}'$, while a union reducible element of $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that there exists a subcovering $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ for which $K = \bigcup \mathbb{C}'$. The equality $\mathbb{C}_1 = \mathbb{C}_\cup$ was established in [141], while the other coverings are different in general. Also, note that $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cap$ are subcoverings of $\mathbb{C}$, while $\mathbb{C}_3$ and $\mathbb{C}_4$ are not. In addition, it holds that $\mathbb{C}_2 \subseteq \mathbb{C}_\cap$ [141].

Next, we discuss generalizations of the subsystem-based definitions of Pawlak's model presented in Eqs. (2.8) and (2.9). Yao and Yao used the notion of a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection [189]. Given a closure system $\mathbb{S}$ over U, one can construct its dual system $\mathbb{S}' = \{K^c \mid K \in \mathbb{S}\}$. The system $\mathbb{S}'$ contains $\emptyset$ and is closed under set union. Given $S = (\mathbb{S}', \mathbb{S})$, a pair of dual lower and upper approximations can be defined as follows: for $X \subseteq U$,

$$\underline{\text{apr}}_S(X) = \bigcup\{K \in \mathbb{S}' \mid K \subseteq X\}, \tag{2.23}$$

$$\overline{\text{apr}}_S(X) = \bigcap\{K \in \mathbb{S} \mid X \subseteq K\}. \tag{2.24}$$

As a particular example of a closure system, Yao and Yao [189] considered the intersection-closure $S_{\cap,\mathbb{C}}$ of a covering $\mathbb{C}$, i.e., the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set intersection:

$$S_{\cap,\mathbb{C}} = \left\{\bigcap\{K \subseteq U \mid K \in F\} \mid F \subseteq \mathbb{C}\right\}. \tag{2.25}$$

Note that the intersection-closure $S_{\cap,\mathbb{C}}$ is a meet-semilattice, i.e., every non-empty finite subset of $S_{\cap,\mathbb{C}}$ has a greatest lower bound in $S_{\cap,\mathbb{C}}$. On the other hand, the union-closure of $\mathbb{C}$, denoted by $S_{\cup,\mathbb{C}}$, is the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set union:

$$S_{\cup,\mathbb{C}} = \left\{\bigcup\{K \subseteq U \mid K \in F\} \mid F \subseteq \mathbb{C}\right\}. \tag{2.26}$$

The union-closure $S_{\cup,\mathbb{C}}$ is a join-semilattice, i.e., every non-empty finite subset of $S_{\cup,\mathbb{C}}$ has a least upper bound in $S_{\cup,\mathbb{C}}$. It can be shown that the dual system $(S_{\cup,\mathbb{C}})'$ forms a closure system. Both systems $S_\cap = ((S_{\cap,\mathbb{C}})', S_{\cap,\mathbb{C}})$ and $S_\cup = (S_{\cup,\mathbb{C}}, (S_{\cup,\mathbb{C}})')$ can be used to obtain two pairs of dual approximation operations, according to Eqs. (2.23) and (2.24).

2.2.2 Framework of Yang and Li

Besides the study of dual generalized approximation operators, various authors have studied upper approximation operators paired with a non-dual lower approximation operator [132, 156, 173, 174, 194, 199, 200]. Yang and Li [177] provided an overview of these non-dual pairs of approximation operators, which is resumed below. Let $(U, \mathbb{C})$ be a covering approximation space, then we can define seven upper approximation operators $H_i^{\mathbb{C}} : \mathscr{P}(U) \to \mathscr{P}(U)$ as follows: let $X \subseteq U$, then

$$H_1^{\mathbb{C}}(X) = \underline{\text{apr}}'_{\mathbb{C}}(X) \cup \left(\bigcup\left\{\bigcup \text{md}(\mathbb{C}, x) \mid x \in X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)\right\}\right) \tag{2.27}$$

$$
\begin{aligned}
H_2^{\mathbb{C}}(X) &= \bigcup\{K \in \mathbb{C} \mid K \cap X \neq \emptyset\} && (2.28)\\
H_3^{\mathbb{C}}(X) &= \bigcup\left\{\bigcup \mathrm{md}(\mathbb{C}, x) \mid x \in X\right\} && (2.29)\\
H_4^{\mathbb{C}}(X) &= \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \cup \left(\bigcup\left\{K \in \mathbb{C} \mid K \cap (X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)) \neq \emptyset\right\}\right) && (2.30)\\
H_5^{\mathbb{C}}(X) &= \bigcup\{N_1^{\mathbb{C}}(x) \mid x \in X\} && (2.31)\\
H_6^{\mathbb{C}}(X) &= \{x \in U \mid N_1^{\mathbb{C}}(x) \cap X \neq \emptyset\} && (2.32)\\
H_7^{\mathbb{C}}(X) &= \bigcup\{N_1^{\mathbb{C}}(x) \mid N_1^{\mathbb{C}}(x) \cap X \neq \emptyset\} && (2.33)
\end{aligned}
$$

The upper approximation operators $H_1^{\mathbb{C}}$ to $H_5^{\mathbb{C}}$ were studied with the tight lower approximation operator $\underline{\mathrm{apr}}'_{\mathbb{C}}$, and the operators $H_6^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ with the tight lower approximation operator for the covering $\mathbb{C}_3$, i.e., $\underline{\mathrm{apr}}'_{\mathbb{C}_3}$. Note that our definition of $H_1^{\mathbb{C}}$ and $H_3^{\mathbb{C}}$ slightly differs from the one presented in [177]: indeed, we need to add one extra $\bigcup$-symbol compared to [177], since the minimal description of x is a set of sets. It is clear from the definition that $H_2^{\mathbb{C}} = \overline{\mathrm{apr}}''_{\mathbb{C}}$ and $H_6^{\mathbb{C}} = \overline{\mathrm{apr}}_{N_1^{\mathbb{C}}}$. Although originally the upper approximation operators were studied with a non-dual lower approximation operator, we will consider the dual pairs $(L_i^{\mathbb{C}}, H_i^{\mathbb{C}})$, with $L_i^{\mathbb{C}}(X) = (H_i^{\mathbb{C}}(X^c))^c$ for $X \subseteq U$.

2.2.3 Framework of Zhao

In [197], Zhao studied covering-based approximation operators from a topological point of view. Therefore, we discuss some essential topological concepts [119].

Definition 2.2.5. [119] A *topology* on a set U is a collection $\mathcal{T}$ of subsets of U which has the following properties:

1. $\emptyset$ and U are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

The ordered pair $(U, \mathcal{T})$ is called a *topological space*. The set $X \subseteq U$ is said to be *open* in U if it belongs to the collection $\mathcal{T}$, and *closed* in U if X^c is open in U. The *interior* $\mathrm{int}(X)$ of X is the union of all open sets in U contained in X. The *closure* $\overline{X}$ of X is the intersection of all closed sets in U containing X.

Furthermore, let $V \subseteq U$, then $(V, \mathscr{T}_V)$ is called a *topological subspace* of $(U, \mathscr{T})$, where

$$\mathscr{T}_V = \{V \cap X \mid X \in \mathscr{T}\}.$$

Moreover, a *separation* of U is a pair (X, Y) of disjoint non-empty open sets in U whose union is U. The space $(U, \mathscr{T})$ is said to be *connected* if there does not exist a separation of U.

Definition 2.2.6. [197] Given a topological space $(U, \mathscr{T})$, we define an equivalence relation on U by setting $x \sim y$ for $x, y \in U$ if there is a connected subspace of $(U, \mathscr{T})$ containing both x and y. For each x in U, the equivalence class of x is called the *component* of x and is denoted by $[x]_\sim$.

Zhao [197] introduced the *topology induced by* $\mathbb{C}$, and showed that this notion indeed satisfies the conditions of Definition 2.2.5.

Definition 2.2.7. [197] Let $(U, \mathbb{C})$ be a covering approximation space. The topology $\mathscr{T}$ on U induced by $\mathbb{C}$ is defined as follows: $X \subseteq U$ is open in U if and only if for each x in X, there exist a subset $\{C_1, C_2, \ldots, C_n\}$ of $\mathbb{C}$ such that $x \in \bigcap_{i=1}^{n} C_i \subseteq X$.

We illustrate the concepts of the previous two definitions in the following example:

Example 2.2.8. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, then the topology induced by $\mathbb{C}$ is given by

$$\mathscr{T} = \{\emptyset, U, \{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}.$$

For $V_1 = \{1\}$ and $V_2 = \{2, 3, 4\}$ it holds that $(V_1, \mathscr{T}_{V_1})$ and $(V_2, \mathscr{T}_{V_2})$ are connected subspaces. Therefore it holds that $U/\sim = \{\{1\}, \{2, 3, 4\}\}$.

Given a covering approximation space $(U, \mathbb{C})$ and the topology $\mathscr{T}$ induced by $\mathbb{C}$, Zhao [197] studied the following seven pairs of dual covering-based approximation operators in $(U, \mathbb{C})$:

- The pair (l^-, l^+) with for $X \subseteq U$,

$$l^-(X) = \{x \in U \mid N_1^{\mathbb{C}}(x) \subseteq X\} \quad (2.34)$$

$$
\begin{aligned}
&= \operatorname{int}(X), \\
l^+(X) &= \{x \in U \mid N_1^{\mathbb{C}}(x) \cap X \neq \emptyset\} \qquad (2.35)\\
&= \overline{X}.
\end{aligned}
$$

- The pair (r^-, r^+) with for $X \subseteq U$,

$$
\begin{aligned}
r^-(X) &= \{x \in U \mid (\forall u \in U)(x \in N_1^{\mathbb{C}}(u) \Rightarrow u \in X)\} \\
&= \{x \in U \mid \overline{\{x\}} \subseteq X\}, \qquad (2.36)\\
r^+(X) &= \bigcup\{N_1^{\mathbb{C}}(x) \mid x \in X\} \\
&= \{x \in U \mid \overline{\{x\}} \cap X \neq \emptyset\}. \qquad (2.37)
\end{aligned}
$$

- The pair (s^-, s^+) with for $X \subseteq U$,

$$
\begin{aligned}
s^-(X) &= \{x \in U \mid N_1^{\mathbb{C}}(x) \subseteq X \vee \overline{\{x\}} \subseteq X\} \qquad (2.38)\\
&= l^-(X) \cup r^-(X), \\
s^+(X) &= \{x \in U \mid N_1^{\mathbb{C}}(x) \cap X \neq \emptyset \wedge \overline{\{x\}} \cap X \neq \emptyset\} \qquad (2.39)\\
&= l^+(X) \cap r^+(X).
\end{aligned}
$$

- The pair (b^-, b^+) with for $X \subseteq U$,

$$
\begin{aligned}
b^-(X) &= \{x \in U \mid N_1^{\mathbb{C}}(x) \cup \overline{\{x\}} \subseteq X\} \qquad (2.40)\\
&= l^-(X) \cap r^-(X), \\
b^+(X) &= \{x \in U \mid (N_1^{\mathbb{C}}(x) \cup \overline{\{x\}}) \cap X \neq \emptyset\} \qquad (2.41)\\
&= l^+(X) \cup r^+(X).
\end{aligned}
$$

- The pair (z^-, z^+) with for $X \subseteq U$,

$$
\begin{aligned}
z^-(X) &= \{x \in U \mid \overline{N_1^{\mathbb{C}}(x)} \subseteq X\}, \qquad (2.42)\\
z^+(X) &= \{x \in U \mid \overline{N_1^{\mathbb{C}}(x)} \cap X \neq \emptyset\}. \qquad (2.43)
\end{aligned}
$$

Recall that $\overline{N_1^{\mathbb{C}}(x)} = \bigcap\{Y \in \mathscr{T}^c \mid N_1^{\mathbb{C}}(x) \subseteq Y\}$, with $\mathscr{T}^c$ the set of closed sets defined by $\{Z^c \mid Z \in \mathscr{T}\}$.

- The pair $(\mathrm{COM}^-, \mathrm{COM}^+)$ with for $X \subseteq U$,

$$\begin{aligned}\mathrm{COM}^-(X) &= \bigcup\{Y \in U/{\sim} \mid Y \subseteq X\}, &(2.44)\\ \mathrm{COM}^+(X) &= \bigcup\{Y \in U/{\sim} \mid Y \cap X \neq \emptyset\}. &(2.45)\end{aligned}$$

- The pair $(\underline{P}_4, \overline{P}_4)$ with for $X \subseteq U$,

$$\begin{aligned}\underline{P}_4(X) &= \bigcup\left\{P_x^{\mathbb{C}} \mid x \in U \wedge P_x^{\mathbb{C}} \subseteq X\right\} &(2.46)\\ &= \{x \in U \mid P_x^{\mathbb{C}} \subseteq X\},\\ \overline{P}_4(X) &= \bigcup\left\{P_x^{\mathbb{C}} \mid x \in U \wedge P_x^{\mathbb{C}} \cap X \neq \emptyset\right\} &(2.47)\\ &= \{x \in U \mid P_x^{\mathbb{C}} \cap X \neq \emptyset\}.\end{aligned}$$

  where the *adhesion* $P_x^{\mathbb{C}}$ of x in U is defined by

$$\begin{aligned}P_x^{\mathbb{C}} &= \{y \in U \mid (\forall K \in \mathbb{C})(x \in K \Leftrightarrow y \in K)\}\\ &= \{y \in U \mid N_1^{\mathbb{C}}(x) = N_1^{\mathbb{C}}(y)\}.\end{aligned}$$

2.2.4 Framework of Samanta and Chakraborty

In [142,143], Samanta and Chakraborty did an extensive study on various covering-based rough set approximation operators. They discussed the following pairs of dual covering-based approximation operators in the covering approximation space $(U, \mathbb{C})$:

- The pair $(\underline{P}_1, \overline{P}_1) = (\underline{\mathrm{apr}}_{N_4^{\mathbb{C}}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}}})$.
- The pair $(\underline{P}_2, \overline{P}_2)$ with for $X \subseteq U$,

$$\begin{aligned}\underline{P}_2(X) &= \bigcup\{N_4^{\mathbb{C}}(x) \mid x \in U, N_4^{\mathbb{C}}(x) \subseteq X\}, &(2.48)\\ \overline{P}_2(X) &= \{x \in U \mid \forall y \in U\colon x \in N_4^{\mathbb{C}}(y) \Rightarrow N_4^{\mathbb{C}}(y) \cap X \neq \emptyset\}. &(2.49)\end{aligned}$$

- The pair $(\underline{P}_3, \overline{P}_3) = (\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$.
- The pair $(\underline{P}_4, \overline{P}_4)$ as defined in Eqs. (2.46) and (2.47).

- The pair $(\underline{C}_1, \overline{C}_1) = (\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$.
- The pair $(\underline{C}_2, \overline{C}_2) = (\underline{\text{apr}}_{N_1^{\mathbb{C}}}, \overline{\text{apr}}_{N_1^{\mathbb{C}}})$.
- The pair $(\underline{C}_3, \overline{C}_3)$ with for $X \subseteq U$,

$$\underline{C}_3(X) = \{x \in U \mid \exists y \in U \colon y \in N_1^{\mathbb{C}}(x) \wedge N_1^{\mathbb{C}}(y) \subseteq X\}, \quad (2.50)$$
$$\overline{C}_3(X) = \{x \in U \mid \forall y \in U \colon y \in N_1^{\mathbb{C}}(x) \Rightarrow N_1^{\mathbb{C}}(y) \cap X \neq \emptyset\}. \quad (2.51)$$

- The pair $(\underline{C}_4, \overline{C}_4) = (L_7^{\mathbb{C}}, H_7^{\mathbb{C}})$.
- The pair $(\underline{C}_5, \overline{C}_5)$ with for $X \subseteq U$,

$$\underline{C}_5(X) = \{x \in U \mid \forall y \in U \colon x \in N_1^{\mathbb{C}}(y) \Rightarrow y \in X\}, \quad (2.52)$$
$$\overline{C}_5(X) = \bigcup\{N_1^{\mathbb{C}}(x) \mid x \in X\}. \quad (2.53)$$

Moreover, Samanta and Chakraborty considered the following upper approximation operators together with the lower approximation operator $\underline{\text{apr}}'_{\mathbb{C}}$:

- The operator $\overline{C}^* = H_1^{\mathbb{C}}$.
- The operator $\overline{C}^- = H_2^{\mathbb{C}}$.
- The operator $\overline{C}^\# = H_3^{\mathbb{C}}$.
- The operator $\overline{C}^@ = H_4^{\mathbb{C}}$.
- The operator $\overline{C}^+$ defined by, for $X \subseteq U$,

$$\overline{C}^+(X) = \underline{\text{apr}}'_{\mathbb{C}}(X) \cup \left(\bigcup\{N_1^{\mathbb{C}}(x) \mid x \in X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)\}\right). \quad (2.54)$$

  Note that our definition of $\overline{C}^+$ slightly differs from the one presented in [143]: indeed, we need to add one extra $\bigcup$-symbol compared to [143], since $N_1^{\mathbb{C}}(x)$ is a set.

- The operator $\overline{C}^{\%}$ defined by, for $X \subseteq U$,

$$\begin{aligned} \overline{C}^{\%}(X) &= \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \cup \\ & \left(\bigcup\left\{\bigcup\{N_4^{\mathbb{C}}(y) \mid y \in U \setminus N_4^{\mathbb{C}}(x)\} \mid x \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\right\}\right)^{\mathrm{c}}. \end{aligned} \tag{2.55}$$

We will study these six upper approximation operators $\overline{C}^i$ with their dual lower approximation operator $\underline{C}^i$ for $i \in \{*, -, \#, @, +, \%\}$ defined for $X \subseteq U$ by

$$\underline{C}^i(X) = (\overline{C}^i(X^{\mathrm{c}}))^{\mathrm{c}}.$$

Finally, Samanta and Chakraborty studied the pair $(\underline{C}^{Gr}, \overline{C}^{Gr})$ with for $X \subseteq U$,

$$\underline{C}^{Gr}(X) = \bigcup\{K \in \mathbb{C} \mid K \subseteq X\}, \tag{2.56}$$

$$\overline{C}^{Gr}(X) = \left(\bigcup\{K \in \mathbb{C} \mid K \cap X \neq \emptyset\}\right) \setminus \left(\underline{C}^{Gr}(X^{\mathrm{c}})\right). \tag{2.57}$$

2.2.5 Singleton, subset and concept approximation operators

In literature, authors sometimes use the terminology of singleton, subset and concept approximation operators [56, 62]. We will illustrate them by means of the neighborhood operator N. Given $X \subseteq U$, the singleton approximations of X are given by

$$\begin{aligned} \underline{\mathrm{apr}}_{\mathrm{sing}}(X) &= \{x \in U \mid \mathrm{N}(x) \subseteq X\}, \\ \overline{\mathrm{apr}}_{\mathrm{sing}}(X) &= \{x \in U \mid \mathrm{N}(x) \cap X \neq \emptyset\}, \end{aligned}$$

i.e., the singleton approximation operators coincide with the element-based approximation operators $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$. The subset approximations of X are given by

$$\begin{aligned} \underline{\mathrm{apr}}_{\mathrm{subs}}(X) &= \bigcup\{\mathrm{N}(x) \mid x \in U, \mathrm{N}(x) \subseteq X\}, \\ \overline{\mathrm{apr}}_{\mathrm{subs}}(X) &= \bigcup\{\mathrm{N}(x) \mid x \in U, \mathrm{N}(x) \cap X \neq \emptyset\}, \end{aligned}$$

i.e., the subset approximation operators coincide with the non-dual pair of granule-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}})$ when N is reflexive and the covering

$\mathbb{C} = \{\mathrm{N}(x) \mid x \in U\}$ is considered. Finally, the concept approximations of X are given by

$$\begin{aligned}
\underline{\mathrm{apr}}_{\mathrm{conc}}(X) &= \bigcup\{\mathrm{N}(x) \mid x \in X, \mathrm{N}(x) \subseteq X\}, \\
&= \underline{\mathrm{apr}}_{\mathrm{subs}}(X) \\
\overline{\mathrm{apr}}_{\mathrm{conc}}(X) &= \bigcup\{\mathrm{N}(x) \mid x \in X, \mathrm{N}(x) \cap X \neq \emptyset\}, \\
&= \bigcup\{\mathrm{N}(x) \mid x \in X\}.
\end{aligned}$$

Therefore, both $H_3^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ can be seen as concept upper approximation operators by taking $\mathrm{N} = N_2^{\mathbb{C}}$ and $\mathrm{N} = N_1^{\mathbb{C}}$ respectively. We will prove in Chapter 4 that $\overline{\mathrm{apr}}_{\mathrm{conc}}$ is equivalent to an element-based upper approximation operator. Therefore, we shall not consider this categorization of approximation operators in the remainder of this dissertation, as the singleton, subset and concept approximation operators can be described by element-based and granule-based approximation operators.

2.3 Variable precision rough set model

The original model designed by Pawlak has strict definitions, in the sense that it does not allow misclassification: changing one element can lead to drastic changes in the lower and upper approximation. The Variable Precision Rough Set (VPRS) model proposed by Ziarko in 1993 [203] is designed to include tolerance to noisy data. In this model, we allow some misclassification. To do this, we generalize the standard set inclusion.

Let X and Y be non-empty subsets of the universe U. In the classical definition of set inclusion, there is no room for misclassification, i.e., X is only included in Y if all elements of X belong to Y. There is no distinction between sets that are more included in Y than others. We introduce the measure to evaluate the *relative degree of misclassification* of a set X with respect to a set Y:

$$c(X,Y) = \begin{cases} 1 - \frac{|X \cap Y|}{|X|} & \text{if } X \neq \emptyset \\ 0 & \text{if } X = \emptyset \end{cases} \tag{2.58}$$

where $|X|$ denotes the cardinality of the set X. We also call $c(X,Y)$ the relative classification error and $c(X,Y) \cdot |X|$ the absolute classification error. The more elements X and Y have in common, the lower the relative degree of misclassification. So, if X is included in Y according to the classical definition of inclusion, then $c(X,Y) = 0$. Based on the measure $c(X,Y)$, we can characterize the classical inclusion of X in Y without explicitly using a quantifier:

$$X \subseteq Y \Leftrightarrow c(X,Y) = 0.$$

We can extend this in a natural way to the majority inclusion relation [203]. Given $0 \leq \beta < 0.5$ and $X,Y \subseteq U$, the *majority inclusion relation* between X and Y is defined as

$$X \overset{\beta}{\subseteq} Y \Leftrightarrow c(X,Y) \leq \beta. \tag{2.59}$$

We obtain the standard set inclusion (or total inclusion) for $\beta = 0$.

In addition, let R be a binary relation on U. For $X \subseteq U$ and $x \in U$ we define the *rough membership function* R_X of X as

$$R_X(x) = 1 - c(R^p(x),X) = \begin{cases} \frac{|R^p(x)\cap X|}{|R^p(x)|} & R^p(x) \neq \emptyset \\ 1 & R^p(x) = \emptyset \end{cases} \tag{2.60}$$

The rough membership $R_X(x)$ quantifies the degree of inclusion of $R^p(x)$ into X and can be interpreted as the conditional probability that x belongs to X, given its foreset $R^p(x)$.

Ziarko worked in a Pawlak approximation space, but we can also introduce the model in a relation approximation space (U,R). We work with asymmetric boundaries as proposed by Katzberg and Ziarko.

Definition 2.3.1. [84] Let (U,R) be a relation approximation space, then the variable precision rough set approximation operators $(\underline{\text{apr}}_{\text{R,u}}, \overline{\text{apr}}_{\text{R,l}})$ for $0 \leq l < u \leq 1$ are defined for $X \subseteq U$ by

$$\underline{\text{apr}}_{\text{R,u}}(X) = \{x \in U \mid R_X(x) \geq u\}, \tag{2.61}$$

$$\overline{\text{apr}}_{\text{R,l}}(X) = \{x \in U \mid R_X(x) > l\}. \tag{2.62}$$

When $u = 1 - l$, the rough set model presented in Definition 2.3.1 is called a symmetric VPRS model. The original VPRS model proposed by Ziarko was based on an equivalence relation E and assumed $0 \leq l < 0.5$ and $u = 1 - l$. With $u = 1$, $l = 0$ and R an equivalence relation, we obtain the original rough set model of Pawlak.

Although the VPRS model of Ziarko is more tolerant to noisy data in comparison with the model of Pawlak, it lacks many properties of the latter. For $0 < l < u < 1$, the VPRS model satisfies properties (SM) and (UE). When $0 \leq l < 0.5$ and $u = 1 - l$, the model also satisfies the duality property (D).

CHAPTER 3

Semantical approach to rough set theory

In a recent paper, Yao [186] argued that there are two sides to rough set theory: a conceptual and a computational one. In a *conceptual approach* it is studied how to define various notions and concepts of the theory, while in a *computational approach* it is studied how to compute them. Therefore, the former approach provides insights in the concepts of the theory, but may not supply computationally efficient algorithms, whereas the latter approach is very suitable for computations and applications, but the meaning of the concepts may be lost. Hence, both approaches are fundamental in the research on rough set theory.

A fundamental task of rough set theory is to analyze data representation in order to derive decision rules [60]. The left-hand-side (LHS) and the right-hand-side (RHS) of a rule are descriptions of two concepts and the rule is a linkage between the two concepts. In general, the left-hand-side consists of a conjunction of atomic formulas (atoms), where an *atomic formula* describes the smallest information block for a given attribute of the table and a possible value of that attribute. The right-hand-side of a rule consists of a disjunction of such atomic formulas. If an object satisfies all the atomic formulas in the left-hand-side of a decision rule, it will satisfy one of the atomic formulas in the right-hand-side of the rule. Hence, we can make a decision on this object.

It is important to have a formal way to represent and interpret those descriptors. To describe the semantics of a concept, we discuss its *intension* and its *extension* [186]. While the intension of a concept describes the properties which are characteristics of the concept, the extension of a concept contains all the objects satisfying the properties of the intension. Unfortunately, the intensions of concepts are barely discussed in the computational models, and if they are discussed, the intension and extension of a concept are not explicitly connected, as happens in Bonikowski et al. [8].

Such a semantically sound approach using both the intension and extension was already suggested by Pawlak [126] and Marek and Pawlak [111] prior to the introduction of rough set theory. However, except for a few articles by Marek and Truszczyński [110] and Yao [185, 187, 191], the conceptual formulation of rough sets is scarcely discussed. For three decades, the focus of the rough set research field has been on the computational approach of rough set approximation operators, as we will discuss in Chapter 4. However, whereas in the model of Pawlak there is a clear semantical connection between the given data in the information or decision table, this connection is often absent in generalized models.

With the aim of refocussing our attention again on the earlier research, we recall the rough set framework of Pawlak [128, 129] from a semantical point of view. Starting from the data, the *definability* of subsets of the universe is discussed before the notion of approximation operators [185]. In order to do this, a *descriptive language* is constructed in two parts. The formulas in the language are considered the intensions of the concepts. Corresponding to a formula, its *meaning set*, i.e., the set of objects satisfying the formula, is the extension of the concept. A set of objects is therefore definable, if it is the meaning set of a formula in the descriptive language, otherwise, it is undefinable. From this point of view, approximation operators are introduced in order to describe undefinable sets by means of definable sets. Given an undefinable set X, the largest definable set contained in X is called the lower approximation of X, while the smallest definable set containing X is called the upper approximation of X. Therefore, these approximation operators are the only meaningful ones in this framework. Moreover, in Section 3.3.4 it will be discussed that the definable sets for Pawlak's model can be computed by a Boolean algebra over a partition related to the data.

Unfortunately, it is not always possible to define such a partition. For instance, in ordered information tables, in which the equivalence classes will mostly consist of only one object, it is unreasonable for applications such as rule induction to construct a partition. For analyzing such information tables, Greco et al. used dominance-based rough sets [53, 54, 149]. Other examples include the computational approaches for incomplete information tables discussed by Kryszkiewicz [88, 89] and Grzymala-Busse [59]. Therefore, we extend the semantical approach of Pawlak's model to covering-based rough set models. However, the definable sets will no longer be computed by the use of a Boolean algebra over a partition, but by the union-closure of a covering, which is a join-semilattice.

The outline of this chapter is as follows. First, we give a formal overview of the results obtained in Sections 3.2 and 3.3. Next, we recall a semantical approach for Pawlak's rough set model, for which we construct a two-part language. Furthermore, we introduce a semantical approach for covering-based rough set models. We apply this semantical approach to dominance-based rough sets in Section 3.4. In addition, we introduce a semantical approach for decision tables with missing values using Pawlak's rough set model in Section 3.5. We end with conclusions and future work in Section 3.6.

3.1 A new conceptual understanding of rough set models

A possible application of Pawlak's rough set model and covering-based rough set models is rule induction. It is an important technique to extract knowledge from data represented in a decision table [60]. Here, we assume the decision table T to be complete with T the tuple

$$T = (U, At = C \cup \{d\}, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

where U is a finite non-empty set of objects, At is a finite non-empty set of attributes consisting of the set of conditional attributes C and the decision attribute d, V_a

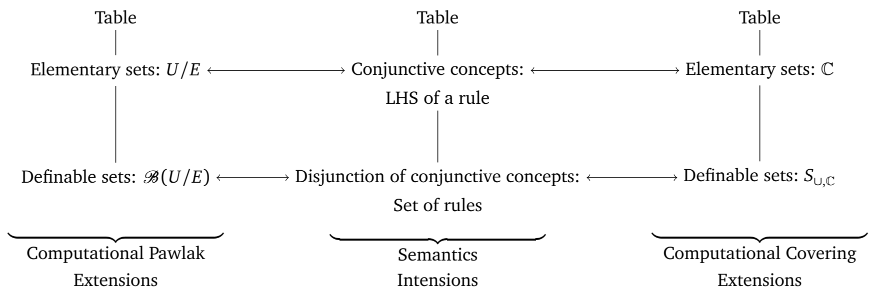


Figure 3.1: Scheme comparing Pawlak's model and covering-based models for a complete decision table

is the non-empty domain of $a \in At$, and $I_a : U \to V_a$ is a complete information function for $a \in At$ which maps every object to exactly one value in V_a.

Knowledge in a decision table may be described with a set of rules, where each rule consists of a condition part (left-hand-side) and decision part (right-hand-side), based on the conditional and decision attributes of the table, respectively. In general, the condition part of a rule can be written as a conjunction of atoms, while the decision part of the rule consists of a disjunction of atoms. For example, a rule can be represented as follows:

$$\text{If object } x \text{ satisfies condition}_1 \wedge \text{condition}_2 \wedge \ldots \wedge \text{condition}_n,$$
$$\text{then } x \text{ satisfies decision}_1 \vee \text{decision}_2 \vee \ldots \vee \text{decision}_m.$$

Every atom is a formula related to an attribute a and one of its values $v \in V_a$. The set consisting of objects related with such an atomic formula is the smallest indivisible block of information we can obtain from a decision table given the pair (a, v). Every object which satisfies the condition part of a rule, will certainly or possibly satisfy the decision part of that rule, depending whether the rule is certain or possible [64]. We will restrict to certain rules, i.e., if an objects satisfies all the information blocks in the condition part, it will satisfy at least one of the information blocks in the decision part. Therefore, rule induction can be used for the classification of new instances.

In Figure 3.1, a schematic overview is given of the results which we shall present in Sections 3.2 and 3.3. Given a table representing the data, a descriptive language is constructed. The formulas of this descriptive language represent the intensions of the considered concepts ('Semantics'). The formulas of the language are constructed in two parts. First, we describe formulas consisting of the conjunction of atomic formulas of the language. Concepts with a conjunctive formula as intension are called *conjunctive concepts*. Next, the disjunctions of conjunctive formulas are described.

Moreover, the extensions of the considered concepts are discussed. These extensions are represented by a subset of the universe of discourse. The extension corresponding to a conjunctive formula is called an elementary set, while a disjunction of conjunctive formulas results in a definable set. The elementary and definable sets can be regarded as a certain structure related to the universe which depends on the chosen framework (Pawlak or covering-based).

Given the family of definable sets, we are able to define approximation operators based on this family. As every definable set can be regarded as a union of elementary sets, every subset X of the universe of discourse can either be described by elementary sets if X is definable, or X can be approximated by elementary sets if it is undefinable.

In Section 3.2, we will describe a semantically sound approach for Pawlak's model. We show that the elementary sets can be computationally described by a partition U/E on the universe, determined by an equivalence relation E [128]. Moreover, the definable sets are formed by closing the partition under union, resulting in a Boolean algebra $\mathscr{B}(U/E)$ over this partition. Although Pawlak's model provides very strong structures to describe the elementary and definable sets, it is not always possible to build an equivalence relation (see e.g., [53, 54, 59, 88, 149]). Therefore, the descriptive language associated with Pawlak's model should be generalized. As we will discuss in Section 3.3, the elementary sets are generally described by a covering $\mathbb{C}$ instead of a partition U/E. Moreover, the union-closure $S_{\cup,\mathbb{C}}$ of the covering $\mathbb{C}$ will describe the definable sets for the extended generalized descriptive language. In general, this will no longer be a Boolean algebra but a join-semilattice.

3.2 Semantically sound approach of Pawlak's rough set model

We first elaborate on a semantically sound approach of Pawlak's rough set model. We construct a two-part language and discuss the elementary and definable sets corresponding to this language. Moreover, we study how to derive the approximation operators and we discuss how the computational model of Pawlak is related to this semantical approach.

3.2.1 A descriptive language for conjunctive concepts

In order to define the intension of a concept in Pawlak's rough set model, we present the following semantically sound approach. First, we introduce the descriptive language DL for conjunctive concepts.

Definition 3.2.1. The symbols of the language DL are the symbols '=', '(', ')', ',' and '$\wedge$', the finite set of attribute symbols At and the finite set of values V_a for each attribute $a \in At$. The *descriptive language* DL is now defined by

1. atomic formulas or atoms $(a, =, v)$ with $a \in At$ and $v \in V_a$,
2. if $\varphi, \psi \in \mathrm{DL}$, then $(\varphi \wedge \psi) \in \mathrm{DL}$.

That is, the descriptive language DL consists of the atoms $(a, =, v)$ and is closed under finite conjunctions. For $A \subseteq At$, we denote DL_A for the descriptive language[1] restricted to the attributes of A. For the remainder of this section, we will work with the language DL_A, as DL is a special case for $A = At$.

We now define a satisfiability relation on $U \times \mathrm{DL}_A$.

Definition 3.2.2. Let $x \in U$ and $\varphi \in \mathrm{DL}_A$ a formula, *the satisfiability of* φ *by* x *in* DL_A, denoted by $x \vDash_A \varphi$, is defined as follows:

1. $x \vDash_A (a, =, v)$ if and only if $I_a(x) = v$ for $a \in A$ and $v \in V_a$,

[1] For the remainder of this chapter we will use the subscript a instead of $\{a\}$ if $A = \{a\}$ to avoid notation overload. E.g., we denote DL_a instead of $\mathrm{DL}_{\{a\}}$.

2. $x \models_A (\varphi \wedge \psi)$ if and only if $x \models_A \varphi$ and $x \models_A \psi$.

Thus, an object x satisfies an atomic formula $(a, =, v)$ if the information function I_a maps x to v. Moreover, x satisfies the conjunction of formulas, if it satisfies all formulas in the conjunction.

Based on the satisfiability relation $\models_A$ for the descriptive language DL_A, we define the meaning set corresponding to a formula $\varphi \in \mathrm{DL}_A$ as the set of objects in U which satisfy φ:

Definition 3.2.3. Let $\varphi \in \mathrm{DL}_A$ a formula, then the *meaning set* $m_A(\varphi)$ of φ is given by $m_A(\varphi) = \{x \in U \mid x \models_A \varphi\}$.

The meaning set of an atom $(a, =, v)$ consists of those objects x such that $I_a(x) = v$. For $a \in A$, it holds that $m_A((a, =, v)) = m_a((a, =, v))$, for $v \in V_a$. Furthermore, note that the meaning set of a conjunction of formulas equals the intersection of the meaning sets of the formulas in the conjunction. The meaning set $m_A(\varphi)$ is exactly the extension of the concept which intension is the formula φ. A formula of which the meaning set is the whole universe U is called *a valid formula for the language* DL_A.

We can now define the definability of a set of objects for the language DL_A.

Definition 3.2.4. Given the language DL_A, a subset $X \subseteq U$ is called *an A-definable set* if there exists a formula $\varphi \in \mathrm{DL}_A$ such that X is the meaning set of φ, i.e., $X = m_A(\varphi)$. If X is not A-definable, it is called *an A-undefinable set*. The set of A-definable sets given the table T is denoted by $\mathrm{DEF}_{\mathrm{DL}_A}(T)$.

Next, we discuss an important subset of $\mathrm{DEF}_{\mathrm{DL}_A}(T)$. Let φ be the conjunction of information blocks from one row in the table, then φ can serve as the left-hand-side of a rule related to the decision table. Hence, there exists a set $A \subseteq C$, i.e., a set of conditional attributes, and an object $x \in U$ such that

$$\varphi = \bigwedge_{a \in A} (a, =, I_a(x)),$$

i.e., φ is the conjunction over the conditional attributes of A of atoms related to a specific object x. The meaning set of such a formula φ is called *an A-elementary* *set* [128, 186]. Note that every A-elementary set is also A-definable for the language DL_A, but is specifically in correspondence with a formula which represents the left-hand-side of a rule. Therefore, the A-elementary sets will be important from a computational point of view, as they are useful for the classification of a new object: if a new object belongs to an A-elementary set, then it satisfies the left-hand-side of a rule, and thus it certainly satisfies its right-hand-side and hence, it can be classified.

However, given the language DL_A, we are not able to describe the right-hand-side of rules, as they are generally described by a disjunction of atomic formulas in DL_d, with d the decision attribute. Therefore, we have to extend the descriptive language.

3.2.2 A descriptive language for disjunctions of conjunctive concepts

The extended descriptive language is defined as follows:

Definition 3.2.5. The *extended descriptive language* EDL_A for $A \subseteq At$ has the same symbols as the language DL_A, extended with the symbol '$\vee$'. The formulas of EDL_A are defined by

1. if $\varphi \in DL_A$, then $\varphi \in EDL_A$,
2. if $\varphi, \psi \in EDL_A$, then $(\varphi \vee \psi) \in EDL_A$.

Hence, the descriptive language DL_A is included in the extended descriptive language EDL_A. Furthermore, EDL_A is closed under finite disjunctions.

The satisfiability relation $\models_A$ is extended to a satisfiability relation on $U \times EDL_A$:

Definition 3.2.6. Let $x \in U$ and $\varphi \in EDL_A$ a formula, then *the satisfiability of* φ *by* x *in* EDL_A, denoted by $x \models_A^E \varphi$, is defined in the following way:

1. if $\varphi \in DL_A$, then $x \models_A^E \varphi$ if and only if $x \models_A \varphi$,
2. $x \models_A^E (\varphi \vee \psi)$ if and only if $x \models_A^E \varphi$ or $x \models_A^E \psi$.

We see that the satisfiability relation $\models_A^E$ reduces to the relation $\models_A$ for formulas in DL_A. Furthermore, an object x satisfies a disjunction of formulas if it satisfies at least one formula in the disjunction.

For a formula $\varphi \in \mathrm{EDL}_A$, we can define its meaning set as follows:

Definition 3.2.7. The *meaning set* $m_A^E(\varphi)$ of a formula $\varphi \in \mathrm{EDL}_A$ is given by $m_A^E(\varphi) = \{x \in U \mid x \models_A^E \varphi\}$.

Hence, it represents all the objects which satisfy φ for $\models_A^E$. Since the satisfiability relation $\models_A^E$ reduces to the relation $\models_A$ for a formula in DL_A, the meaning set of a formula of DL_A does not change for the extended descriptive language EDL_A. Moreover, note that the meaning set of a disjunction of formulas is the union of the meaning sets of the formulas in the disjunction. If $m_A^E(\varphi) = U$, then φ is called *a valid formula in the language* EDL_A. Note that every formula valid in DL_A is also valid in EDL_A.

We now define the set of definable sets of the language EDL_A:

Definition 3.2.8. The A-definable sets of the table T for the extended descriptive language EDL_A, denoted by $\mathrm{DEF}_{\mathrm{EDL}_A}(T)$, are the subsets $X \subseteq U$ which are the meaning set of a formula in EDL_A, i.e., X is *A-definable* if it is the extension of a concept which intension is a formula in EDL_A. If X is not A-definable, it is called *A-undefinable in* EDL_A.

We are now able to describe both the left-hand-side and the right-hand-side of a rule, as they are formulas in the language EDL_{At}. More specifically, the left-hand-side of the rule is a formula in the language DL_A, with $A \subseteq C$ a set of conditional attributes, and the right-hand-side of the rule is a formula in the language EDL_d, with d the decision attribute.

3.2.3 Approximations of undefinable sets

For an A-undefinable set, we are not able to find a formula in EDL_A. However, we are able to approximate it by A-definable sets from $\mathrm{DEF}_{\mathrm{EDL}_A}(T)$, which leads us to the notion of approximation operators, as discussed by Yao [186, 190]. Naturally, a set $X \subseteq U$ is approximated from below by the family

$$\{Y \in \mathrm{DEF}_{\mathrm{EDL}_A}(T) \mid Y \subseteq X \wedge \forall Z \in \mathrm{DEF}_{\mathrm{EDL}_A}(T), Z \subseteq X : Y \subseteq Z \Rightarrow Y = Z\},$$

which is the family of maximal definable sets contained by X and it is approximated from above by the family of the minimal definable sets containing X:

$$\{Y \in \mathrm{DEF}_{\mathrm{EDL}_A}(T) \mid X \subseteq Y \wedge \forall Z \in \mathrm{DEF}_{\mathrm{EDL}_A}(T), X \subseteq Z : Z \subseteq Y \Rightarrow Y = Z\}.$$

However, in Pawlak's framework, there is a unique maximal definable set contained by X and a unique minimal definable set containing X [190]. Hence, the lower and upper approximation of a set $X \subseteq U$ can be defined as follows:

Definition 3.2.9. Let $X \subseteq U$ and $A \subseteq At$, then the *lower and upper approximation of X in* EDL_A, denoted by $\underline{\mathrm{apr}}_A(X)$ and $\overline{\mathrm{apr}}_A(X)$, are defined as follows:

$$\begin{aligned}
\underline{\mathrm{apr}}_A(X) &= \text{the largest definable set in } \mathrm{EDL}_A \text{ contained by } X,\\
\overline{\mathrm{apr}}_A(X) &= \text{the smallest definable set in } \mathrm{EDL}_A \text{ containing } X.
\end{aligned}$$

In addition, we can define the positive, negative and boundary region of a set $X \subseteq U$:

$$\begin{aligned}
\mathrm{POS}_A(X) &= \text{the largest definable set in } \mathrm{EDL}_A \text{ contained by } X,\\
\mathrm{NEG}_A(X) &= \text{the largest definable set in } \mathrm{EDL}_A \text{ contained by } X^{\mathrm{c}},\\
\mathrm{BND}_A(X) &= (\mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X))^{\mathrm{c}}.
\end{aligned}$$

Note that the correspondence between the two-way and three-way approximation operators stated in Section 2.1 holds for the operators defined above.

If X is A-definable, we see that both the lower and the upper approximation of X are the set X itself. Moreover, the positive region of X is X, the negative region of X is its complement X^{c} and the boundary region of X is empty. Hence, for an A-definable set X, all elements of the universe can be classified either in the positive or negative region of the set X.

To end this section, we discuss the meaning set of the left-hand-side of a rule and the meaning set for a set of rules in detail. This will provide us with a computational approach to Pawlak's model, consistent with the semantically sound approach described above.

3.2.4 Computational approach of Pawlak's rough set model

In [128] it is discussed that the partition U/E and the Boolean algebra $\mathscr{B}(U/E)$ are very important structures to construct the Pawlak approximation operators. Here we will see that both structures are obtained when computing the elementary and definable sets of the descriptive language [190].

Given a set $A \subseteq At$, a set $X \subseteq U$ and a formula $\varphi \in \mathrm{EDL}_A$, the concept (X, φ) with extension X and intension φ is A-definable for the language EDL_A if and only if $X = m_A^E(\varphi)$. As φ is a formula in EDL_A, it is a disjunction of conjunctions of atoms, i.e., $\varphi = (\ldots((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \ldots \vee \varphi_n)$ with each $\varphi_i \in \mathrm{DL}_A$ of the form

$$(\ldots(((a_{ij_1}, =, v_{ij_1}) \wedge (a_{ij_2}, =, v_{ij_2})) \wedge (a_{ij_3}, =, v_{ij_3})) \wedge \ldots \wedge (a_{ij_{m_i}}, =, v_{ij_{m_i}})).$$

Hence, the meaning set of φ is

$$m_A^E(\varphi) = \bigcup_{i=1}^{n} \bigcap_{j=j_1}^{j_{m_i}} m_A((a_{ij}, =, v_{ij})).$$

For $i \in \{1, \ldots, n\}$ and $j \in \{j_1, \ldots, j_{m_i}\}$ the meaning set of $(a_{ij}, =, v_{ij})$ equals the equivalence class $[x_{ij}]_{E_{a_{ij}}}$, with

$$E_{a_{ij}} = \{(y, z) \in U \times U \mid I_{a_{ij}}(y) = I_{a_{ij}}(z)\}$$

and $x_{ij} \in U$ such that $I_{a_{ij}}(x_{ij}) = v_{ij}$. Now, for every $i \in \{1, 2, \ldots, n\}$, we either have that $\bigcap_{j=j_1}^{j_{m_i}} [x_{ij}]_{E_{a_{ij}}}$ is empty or there exists an object $x_i \in U$ and a subset of attributes $A_i \subseteq A$ such that it equals the equivalence class $[x_i]_{E_{A_i}}$ with $E_{A_i} = \{(y, z) \in U \times U \mid \forall a \in A_i \colon I_a(y) = I_a(z)\}$. Let $k \in \{1, 2, \ldots, n\}$ such that $m_A(\varphi_i) = [x_i]_{E_{A_i}}$ for $1 \leq i \leq k$ and $m_A(\varphi_i) = \emptyset$ for $k < i \leq n$, then

$$m_A^E(\varphi) = \bigcup_{i=1}^{k} [x_i]_{E_{A_i}}.$$

Hence, we conclude that the meaning set of a conjunctive formula φ_i, i.e., an A_i-elementary set, is an equivalence class $[x_i]_{E_{A_i}}$ and the meaning set of φ, i.e., an A-definable set, is the union of equivalence classes based on attributes in A, i.e., $m_A^E(\varphi) \in \mathscr{B}(U/E_A)$, where $\mathscr{B}(U/E_A)$ is the Boolean algebra over the partition U/E_A. The Boolean algebra $\mathscr{B}(U/E_A)$ contains $\emptyset$ and U/E_A and is closed under union. However, since U/E_A is a partition, $\mathscr{B}(U/E_A)$ is also closed under intersection and set complement. Therefore, we conclude that the A-definable sets for EDL_A are efficiently computed by constructing $\mathscr{B}(U/E_A)$.

Furthermore, let $X \subseteq U$ be A-undefinable for $A \subseteq At$, then we can approximate X with sets in $\mathrm{DEF}_{\mathrm{EDL}_A}(T) = \mathscr{B}(U/E_A)$. The lower approximation of X is given by the largest definable set in EDL_A which is contained in X. As $\mathscr{B}(U/E_A)$ is closed under union, the lower approximation of X is given by

$$\underline{\mathrm{apr}}_A(X) = \bigcup\{Y \in \mathscr{B}(U/E_A) \mid Y \subseteq X\}.$$

Analogously, as the upper approximation of X is the smallest definable set in EDL_A which contains X and as $\mathscr{B}(U/E_A)$ is closed under intersection, we obtain that

$$\overline{\mathrm{apr}}_A(X) = \bigcap\{Y \in \mathscr{B}(U/E_A) \mid X \subseteq Y\}.$$

Moreover, as $\mathscr{B}(U/E_A)$ is closed under set complement, we have that the lower and upper approximation operator are dual operators:

$$\forall X \subseteq U : (\underline{\mathrm{apr}}_A(X^c))^c = \overline{\mathrm{apr}}_A(X).$$

Note that the pair $(\underline{\mathrm{apr}}_A, \overline{\mathrm{apr}}_A)$ is in fact given by the subsystem-based definition of Pawlak's model $(\underline{\mathrm{apr}}_E, \overline{\mathrm{apr}}_E)$ for the equivalence relation $E = E_A$. Although the subsystem-based definitions are seldom used for computational purposes, it is clear from this discussion that they provide a semantical meaning to the lower and upper approximation of an A-undefinable set X.

In addition, the positive, negative and boundary region of X are given by

$$\begin{aligned}
\mathrm{POS}_A(X) &= \bigcup\{Y \in \mathscr{B}(U/E_A) \mid Y \subseteq X\},\\
\mathrm{NEG}_A(X) &= \bigcup\{Y \in \mathscr{B}(U/E_A) \mid Y \subseteq X^c\},\\
\mathrm{BND}_A(X) &= (\mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X))^c
\end{aligned}$$

$$= \bigcap\{Y \in \mathscr{B}(U/E_A) \mid Y \cap X \neq \emptyset \wedge Y \cap X^c \neq \emptyset\}.$$

If $X \in \mathrm{DEF}_{\mathrm{EDL}_A}(T)$, then $\mathrm{POS}_A(X) = X$ and $\mathrm{NEG}_A(X) = X^c$, since $\mathscr{B}(U/E_A)$ is closed under set complement. Therefore, the boundary region of an A-definable set in Pawlak's rough set model is empty.

3.3 Semantically sound approach of covering-based rough set models

To discuss a semantically sound approach of covering-based rough set models for a complete decision table, we first determine how the universe can be granulated. A granulation of the objects of the universe given an attribute $a \in At$ is determined by relationships between the attribute values of V_a. Examples of such relationships are the equality relation $=$, partial order relations $\succeq$, the membership relation $\in$, or relationships obtained from clustering. However, other types of relationships than the equality relation do not fit in the approach discussed in Section 3.2 and hence, a generalization is needed.

In order to do this, we add extra semantics to the information or decision table T [183]. Let L_a be a set of labels for an attribute $a \in At$ which is used to name the granules of the attribute value set V_a. In general, every label in L_a can be interpreted by the values of V_a. The complete decision table with added semantics T_+ is the tuple $(T, \{L_a \mid a \in At\})$, with T the original complete decision table.

Given a set of labels L_a for each attribute $a \in At$, various relationships between the attribute values of a can be constructed, depending on the physical meaning of the labels, i.e., a set of relations R_a is considered instead of only the equality relation $=$. Therefore, a crucial point in generalizing the semantically sound approach of Pawlak's rough set model is defining the atoms regarding the labels L_a and the relations R_a. In order to correctly combine labels from L_a and relations from R_a, a Boolean relation T_a, called a *type compatibility relation* [183], is determined as follows: $T_a(r, l)$ is true for $r \in R_a$ and $l \in L_a$, if and only if it is reasonable to apply the relation r on the label l for the attribute $a \in At$.

We illustrate a decision table with added semantics in the following example.

Table 3.1: Decision table T_+ with $\{L_{a_1}, L_{a_2}, L_d\}$ the set of labels

	a_1	a_2	d
x_1	I	medium	1
x_2	II	bad	0
x_3	III	bad	0
x_4	IV	good	1
x_5	V	good	1

Example 3.3.1. Consider the table T_+ presented in Table 3.1. For a_1, let $R_{a_1} = \{\in\}$ and $L_{a_1} = \{l_1 = \{I, II, III\}, l_2 = \{II, III, IV\}, l_3 = \{III, IV, V\}\}$ such that $T_{a_1}(\in, l_i)$ is true for $i \in \{1, 2, 3\}$. Next, for a_2, let $R_{a_2} = \{\geq\}$ and $L_{a_2} = \{\text{bad}, \text{medium}, \text{good}\}$ such that $T_{a_2}(\geq, l)$ is true for all $l \in L_{a_2}$ and good $\geq$ medium $\geq$ bad. Finally, for d we use the equality relation as with Pawlak's rough set model, i.e., $R_d = \{=\}$ and $L_d = \{0, 1\}$ such that $T_d(=, 0)$ and $T_d(=, 1)$ are true.

Note that there are multiple choices for the set of relations and the set of labels. For instance, for the attribute a_2 we could have chosen $R'_{a_2} = \{\in\}$ and $L'_{a_2} = \{k_1 = \{\text{bad}\}, k_2 = \{\text{bad, medium}\}, k_3 = \{\text{bad, medium, good}\}\}$ such that $T_{a_2}(\in, k_i)$ is true for $i \in \{1, 2, 3\}$. The labels in L'_{a_2} represent respectively the instances $x \in U$ which scored at most 'bad', at most 'medium' and at most 'good'. For the remainder of this section we will work with the set of relations R_{a_2} and the set of labels L_{a_2}.

To describe the semantical approach for covering-based rough set models in a complete decision table with added semantics, we construct a two-part descriptive language.

3.3.1 A generalized descriptive language for conjunctive concepts

Given a complete decision table with added semantics, then we can define the following language:

Definition 3.3.2. The symbols of the *generalized descriptive language* GDL are the symbols '(', ')', ',' and '∧', the finite set of attribute symbols At, the finite set of relation or constraint symbols R_a for each attribute $a \in At$ and the finite set of labels L_a for each $a \in At$. The generalized descriptive language GDL with respect to $\{T_a \mid a \in At\}$ is now defined by

1. atomic formulas or atoms (a, r, l) with $a \in At$, $r \in R_a$, $l \in L_a$ and $T_a(r, l)$ true,
2. if $\varphi, \psi \in$ GDL, then $(\varphi \wedge \psi) \in$ GDL.

Hence, as for the language DL, GDL is closed under finite conjunctions. The main difference between the languages DL and GDL is the set of atoms. Note however, that the language DL is a special case of GDL, when $R_a = \{=\}$ and $L_a = V_a$ for each attribute $a \in At$. Again, for a subset $A \subseteq At$, we can restrict the language GDL to the language GDL_A, when only the attributes in A are considered.

We define a satisfiability relation on $U \times \text{GDL}_A$:

Definition 3.3.3. Let $x \in U$ and $\varphi \in \text{GDL}_A$ a formula, then *the satisfiability of* φ *by* x in GDL_A, denoted by $x \vDash_A \varphi$, is defined as follows:

1. $x \vDash_A (a, r, l)$ if and only if $I_a(x) r l$,
2. $x \vDash_A (\varphi \wedge \psi)$ if and only if $x \vDash_A \varphi$ and $x \vDash_A \psi$.

An object x satisfies an atomic formula $(a, r, l) \in \text{GDL}_A$ if the attribute value $I_a(x)$ is related by r to l. In addition, x satisfies a conjunction of formulas in GDL_A if it satisfies all formulas in the conjunction.

Based on this satisfiability relation, we can define the meaning set of a formula $\varphi \in \text{GDL}_A$:

Definition 3.3.4. Let $\varphi \in \text{GDL}_A$ a formula, then the *meaning set* $m_A(\varphi)$ of φ is given by $m_A(\varphi) = \{x \in U \mid x \vDash_A \varphi\}$.

The meaning set of an atomic formula $(a, r, l) \in \text{GDL}_A$ is the set of objects $x \in U$ such that $I_a(x)$ is related to label l by relation r. Moreover, it holds that

$m_A((a,r,l)) = m_a((a,r,l))$. The meaning set of a conjunction of formulas equals the intersection of meaning sets of the formulas in the conjunction. A formula is called *valid in* GDL_A if its meaning set is the whole universe U.

We illustrate the meaning sets of the atomic formulas related with table T_+ presented in Table 3.1 in the following example.

Example 3.3.5. Consider the table T_+ presented in Table 3.1. We compute the meaning sets of the atomic formulas in Table 3.2.

Table 3.2: Meaning sets of the atomic formulas of Table 3.1 for $a \in \{a_1, a_2, d\}$

Atomic formula	Meaning set
$(a_1, \in, l_1)$	$\{x_1, x_2, x_3\}$
$(a_1, \in, l_2)$	$\{x_2, x_3, x_4\}$
$(a_1, \in, l_3)$	$\{x_3, x_4, x_5\}$
$(a_2, \geq, \text{bad})$	U
$(a_2, \geq, \text{medium})$	$\{x_1, x_4, x_5\}$
$(a_2, \geq, \text{good})$	$\{x_4, x_5\}$
$(d, =, 0)$	$\{x_2, x_3\}$
$(d, =, 1)$	$\{x_1, x_4, x_5\}$

Given the generalized descriptive language GDL_A, we can define its definable sets:

Definition 3.3.6. Given the language GDL_A, the set of *A-definable sets* for the table T_+ contains all meaning sets of formulas of GDL_A and is denoted by $\mathrm{DEF}_{\mathrm{GDL}_A}(T_+)$, i.e., $X \in \mathrm{DEF}_{\mathrm{GDL}_A}(T_+)$ if there exists a formula $\varphi \in \mathrm{GDL}_A$ such that $X = m_A(\varphi)$. If $X \subseteq U$ is not A-definable, it is called an *A-undefinable set*.

Similarly as in Pawlak's model, the left-hand-side of a rule related to the table is a conjunction of information blocks from a row of the table. However, these information blocks are now of the form (a, r, l) instead of $(a, =, I_a(x))$, where l can be described by means of $I_a(x)$ for $x \in U$. Let $A \subseteq C$ a subset of conditional attributes and $x \in U$, then φ is of the form

$$\varphi = \bigwedge_{a \in A} (a, r_a, l_a),$$

where $r_a \in R_a$, $l_a \in L_a$ such that l_a can be described by means of $I_a(x)$ and such that $T_a(r_a, l_a)$ is true. The meaning set of φ is called an *A-elementary set*.

Example 3.3.7. Consider table T_+ presented in Table 3.1. The formula

$$\varphi = (a_1, \in l_1) \wedge (a_2, \geq, \text{medium})$$

represents the left-hand-side of a rule related with the object $x_1 \in U$. Its meaning set

$$m_{\{a_1,a_2\}}(\varphi) = \{x_1, x_2, x_3\} \cap \{x_1, x_4, x_5\} = \{x_1\}$$

is a $\{a_1, a_2\}$-elementary set.

As in the semantically sound approach of Pawlak, the language GDL_A ensures the construction of formulas which can be used in the left-hand-side of a rule. However, as the right-hand-side of a rule consists of disjunctions, the generalized descriptive language also needs to be extended.

3.3.2 A generalized descriptive language for disjunctions of conjunctive concepts

For $A \subseteq At$, the extended generalized descriptive language EGDL_A is constructed similarly to the language EDL_A, i.e., we extend GDL_A by closing it under finite disjunctions:

Definition 3.3.8. The *extended generalized descriptive language* EGDL_A has the same symbols as the language GDL_A, extended with the symbol '$\vee$'. The formulas of EGDL_A with respect to $\{T_a \mid a \in At\}$ are defined by

1. if $\varphi \in \text{GDL}_A$, then $\varphi \in \text{EGDL}_A$,
2. if $\varphi, \psi \in \text{EGDL}_A$, then $(\varphi \vee \psi) \in \text{EGDL}_A$.

We extend the satisfiability relation $\vDash_A$ to $U \times \text{EGDL}_A$:

Definition 3.3.9. Let $x \in U$ and $\varphi \in \text{EGDL}_A$ a formula, then *the satisfiability of* φ *by* x *in* EGDL_A, denoted by $x \vDash_A^E \varphi$, is defined as follows:

1. if $\varphi \in \text{GDL}_A$, then $x \vDash_A^E \varphi$ if and only if $x \vDash_A \varphi$,
2. $x \vDash_A^E (\varphi \vee \psi)$ if and only if $x \vDash_A^E \varphi$ or $x \vDash_A^E \psi$.

Given the satisfiability relation defined above, we can define the meaning set of a formula $\varphi \in \text{EGDL}_A$:

Definition 3.3.10. Let $\varphi \in \text{EGDL}_A$ a formula, then the *meaning set* $m_A^E(\varphi)$ of φ is given by $m_A^E(\varphi) = \{x \in U \mid x \vDash_A^E \varphi\}$.

Since the satisfiability relation $\vDash_A^E$ reduces to the relation $\vDash_A$ for a formula in GDL_A, the meaning set of a formula of GDL_A does not change for the extended generalized descriptive language EGDL_A. Note that the meaning set of a disjunction of formulas equals the union of the meaning sets of the formulas in the disjunction. Furthermore, a formula φ is called *valid in* EGDL_A if $m_A^E(\varphi) = U$.

We can now define the A-definable sets:

Definition 3.3.11. Given the language EGDL_A, the *A-definable sets* of a table T_+, denoted by $\text{DEF}_{\text{EGDL}_A}(T_+)$, are the meaning sets of formulas in EGDL_A. If a set $X \subseteq U$ is not in $\text{DEF}_{\text{EGDL}_A}(T_+)$, X is called *A-undefinable for* EGDL_A.

The language EGDL_{At} allows us to describe both the conditional and decision part of rules. In particular, the conditional part of a rule is a formula in GDL_A with $A \subseteq C$ a set of conditional attributes and the decision part is a formula in EGDL_d with d the decision attribute.

3.3.3 Approximations of undefinable sets

As in the framework of Pawlak, approximation operators are used to approximate an A-undefinable set $X \subseteq U$ by A-definable sets in $\text{DEF}_{\text{EGDL}_A}(T_+)$, $A \subseteq At$. Although there is still a unique maximal A-definable set contained by X, there is no unique minimal A-definable set containing X as opposed to Pawlak's rough set model [190].

Definition 3.3.12. Let $X \subseteq U$ and $A \subseteq At$, then the *lower and upper approximations of* X *in* EGDL_A, denoted by $\underline{\text{apr}}_A(X)$ and $\overline{\text{apr}}_A(X)$, are defined as follows:

$$\begin{aligned} \underline{\text{apr}}_A(X) &= \text{ the largest definable set in } \text{EGDL}_A \text{ contained by } X, \\ \overline{\text{apr}}_A(X) &= \{Y \in \text{DEF}_{\text{EGDL}_A}(T_+) \mid X \subseteq Y, Y \text{ minimal}\}, \end{aligned}$$

with Y minimal if $\forall Z \in \text{DEF}_{\text{EGDL}_A}(T_+), X \subseteq Z : Z \subseteq Y \Rightarrow Y = Z$.

Hence, the upper approximation of X is not a definable set, but the family of minimal definable sets containing X.

The definitions of the lower and upper approximation operator and the positive, negative and boundary region are similar to the framework of Pawlak's rough set model.

$$\begin{aligned} \text{POS}_A(X) &= \text{ the largest definable set in } \text{EGDL}_A \text{ contained by } X, \\ \text{NEG}_A(X) &= \text{ the largest definable set in } \text{EGDL}_A \text{ contained by } X^c, \\ \text{BND}_A(X) &= (\text{POS}_A(X) \cup \text{NEG}_A(X))^c. \end{aligned}$$

Note that the positive and negative region of X are definable sets. Therefore, it is sometimes more useful to consider them instead of the lower and upper approximation operator, as the latter is not definable.

To end this section, we study how to efficiently compute the elementary and definable sets of a covering-based rough set model. It will be shown that a covering and its union-closure will describe the elementary and definable sets instead of a partition and a Boolean algebra as in Pawlak's rough set model.

3.3.4 Computational approach of covering-based rough set models

The main difference between the semantically sound approach of Pawlak's model and covering-based rough set models is the construction of atomic formulas. Given $a \in At$, then the $\{a\}$-elementary set for the atomic formula $(a, r, l) \in \text{GDL}_a$ is the set $\{x \in U \mid I_a(x) r l\}$. Such a set is an *information block*, as it provides the information on which instances of the universe are related to the label l by the relation r. However, such information blocks are not necessarily disjoint, as it is possible that $m((a,r,l)) \cap m((a,r',l')) \neq \emptyset$, with $(a,r',l') \in \mathrm{GDL}_a$. Therefore, the $\{a\}$-elementary sets will be given by a covering

$$\mathbb{C}_a = \{m_a((a,r,l)) \mid m_a((a,r,l)) \neq \emptyset, (a,r,l) \in \mathrm{GDL}_a\}$$

instead of a partition U/E_a. The construction of the covering $\mathbb{C}_a$ is therefore completely depending on the choices of R_a and L_a.

In addition, let $A \subseteq At$ and $\varphi \in \mathrm{GDL}_A$ be a conjunctive formula which serves as the left-hand-side of a rule. Then

$$\varphi = \bigwedge_{a \in A} (a, r_a, l_a)$$

with $(a, r_a, l_a) \in \mathrm{GDL}_A$ and the meaning set of φ is given by

$$m_A(\varphi) = \bigcap_{a \in A} m_A((a, r_a, l_a)).$$

If $m_A(\varphi)$ is not empty, then this A-elementary set represents an information block for the set of attributes A. The set of non-empty A-elementary sets is also a covering, denoted by $\mathbb{C}_A$.

From the above discussion, we obtain the following characterization of the covering $\mathbb{C}_A$ in terms of the coverings $\{\mathbb{C}_a \mid a \in A\}$:

$$\begin{aligned}
\mathbb{C}_A &= \{K \subseteq U \mid K \neq \emptyset \wedge \exists \varphi \in \mathrm{GDL}_A : K = m_A(\varphi)\} \\
&= \{K \subseteq U \mid K \neq \emptyset \wedge \forall a \in A \exists r \in R_a, \exists l \in L_a \text{ such that } T_a(r,l) \text{ is true} \\
&\qquad\qquad \text{and } K = \bigcap_{a \in A} m_A((a,r,l))\} \\
&= \{K \subseteq U \mid K \neq \emptyset \wedge \forall a \in A \exists K_a \in \mathbb{C}_a : K = \bigcap_{a \in A} K_a\}.
\end{aligned}$$

Note that $m_A((a,r,l)) = m_a((a,r,l))$ is an information block in $\mathbb{C}_a$. Hence, $\mathbb{C}_A$ contains of the non-empty intersections $\bigcap_{a \in A} K_a$, where for each $a \in A$ we take exactly one $K_a \in \mathbb{C}_a$.

Furthermore, let $A \subseteq At$, then an A-definable set is the extension of a disjunction of conjunctive formulas in GDL_A. Therefore, the set of meaning sets of formulas from EGDL_A is obtained by closing the covering $\mathbb{C}_A$ under the union operator, which gives the union-closure $S_{\cup,\mathbb{C}_A}$ as defined in Eqs. (2.26). As $\mathbb{C}_A$ is not a partition in general, the union-closure $S_{\cup,\mathbb{C}_A}$ is not closed under intersection and set complement, in comparison with the Boolean algebra which is obtained when given a partition. However, $S_{\cup,\mathbb{C}_A}$ is a join-semilattice.

To end, we discuss the approximation operators related to this semantically sound approach of covering-based rough sets. Let $A \subseteq At$ and $X \subseteq U$, then

$$\underline{\text{apr}}_A(X) = \bigcup\{K \in S_{\cup,\mathbb{C}_A} \mid K \subseteq X\} = \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X\},$$

since $S_{\cup,\mathbb{C}_A}$ is closed under union. Note that this is the *tight* or *strong* lower approximation operator of X for the covering $\mathbb{C}_A$. However, since the union-closure of a covering is not closed under intersection, there is no unique minimal definable set containing X. Hence, we are not able to give a computationally efficient definition of the upper approximation of X in which the semantics are clear. Nonetheless, we are able to give computationally efficient definitions for the positive and negative region of X:

$$\begin{aligned}\text{POS}_A(X) &= \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X\},\\ \text{NEG}_A(X) &= \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X^c\}.\end{aligned}$$

Note that $S_{\cup,\mathbb{C}_A}$ is not closed under set complement and therefore, the complement of an A-definable set is not necessarily A-definable, hence, $\text{BND}_A(X)$ is not necessarily definable and it can be non-empty for $X \in \text{DEF}_{\text{EGDL}_A}(T_+)$ in the covering-based rough set framework.

To end this section, we illustrate the computational approach of covering-based rough set models with an example.

Example 3.3.13. Consider the table with added semantics T_+ presented in Table 3.1. For the conditional attributes a_1 and a_2 we obtain the following coverings:

$$\begin{aligned}\mathbb{C}_{a_1} &= \{\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_3,x_4,x_5\}\},\\ \mathbb{C}_{a_2} &= \{\{x_1,x_2,x_3,x_4,x_5\},\{x_1,x_4,x_5\},\{x_4,x_5\}\}.\end{aligned}$$

The covering $\mathbb{C}_{\{a_1,a_2\}}$ is now obtained by intersecting the elements of $\mathbb{C}_{a_1}$ and $\mathbb{C}_{a_2}$:

$$\mathbb{C}_{\{a_1,a_2\}} = \{\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_3,x_4,x_5\},\{x_1\},\{x_4\},\{x_4,x_5\}\}.$$

Hence, the C-definable sets are given by

$$\begin{aligned} S_{\cup,\mathbb{C}_{\{a_1,a_2\}}} &= \{\emptyset,\{x_1\},\{x_4\},\{x_1,x_4\},\{x_4,x_5\},\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\\ &\quad \{x_3,x_4,x_5\},\{x_1,x_4,x_5\},\{x_1,x_2,x_3,x_4\},\{x_1,x_3,x_4,x_5\},\\ &\quad \{x_2,x_3,x_4,x_5\},\{x_1,x_2,x_3,x_4,x_5\}\}. \end{aligned}$$

Note that $S_{\cup,\mathbb{C}_{\{a_1,a_2\}}}$ is not closed under set intersection and set complement, as $\{x_1,x_2,x_3\}$ and $\{x_2,x_3,x_4\}$ are elements of $S_{\cup,\mathbb{C}_{\{a_1,a_2\}}}$, but $\{x_2,x_3\}$ and $\{x_1,x_5\}$ are not.

To illustrate the approximation operators, consider the set $\{x_3,x_4\}$. We obtain that

$$\begin{aligned} \underline{\text{apr}}_{\{a_1,a_2\}}(\{x_3,x_4\}) &= \{x_4\}\\ \overline{\text{apr}}_{\{a_1,a_2\}}(\{x_3,x_4\}) &= \{\{x_2,x_3,x_4\},\{x_3,x_4,x_5\}\}\\ \text{POS}_{\{a_1,a_2\}}(\{x_3,x_4\}) &= \{x_4\}\\ \text{NEG}_{\{a_1,a_2\}}(\{x_3,x_4\}) &= \{x_1\}\\ \text{BND}_{\{a_1,a_2\}}(\{x_3,x_4\}) &= \{x_2,x_3,x_5\} \end{aligned}$$

As $\overline{\text{apr}}_{\{a_1,a_2\}}(\{x_3,x_4\})$ is a set of C-definable sets, the upper approximation is more complicated from a computational perspective. Moreover, note that there is not necessarily a connection between the conceptual upper approximation operator and the tight covering-based upper approximation operator when we consider the union or intersection: for $X = \{x_2\}$, we have that

$$\begin{aligned} \overline{\text{apr}}_{\{a_1,a_2\}}(\{x_2\}) &= \{\{x_1,x_2,x_3\},\{x_2,x_3,x_4\}\},\\ \overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_2\}) &= \{x_2\}, \end{aligned}$$

hence, neither the inclusion $\bigcup\overline{\text{apr}}_{\{a_1,a_2\}}(\{x_2,x_3\}) \subseteq \overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_2,x_3\})$ nor the inclusion $\bigcap\overline{\text{apr}}_{\{a_1,a_2\}}(\{x_2,x_3\}) \subseteq \overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_2,x_3\})$ holds. For $X = \{x_1,x_4,x_5\}$, we have that

$$\overline{\text{apr}}_{\{a_1,a_2\}}(\{x_1,x_4,x_5\}) = \{\{x_1,x_4,x_5\}\},$$

$$\overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_1,x_4,x_5\}) \quad = \quad U,$$

thus, neither the inclusion $\bigcup \overline{\text{apr}}_{\{a_1,a_2\}}(\{x_1,x_4,x_5\}) \supseteq \overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_1,x_4,x_5\})$ nor the inclusion $\bigcap \overline{\text{apr}}_{\{a_1,a_2\}}(\{x_1,x_4,x_5\}) \supseteq \overline{\text{apr}}'_{\mathbb{C}_{\{a_1,a_2\}}}(\{x_1,x_4,x_5\})$ holds.

We now derive certain rules from the table T_+ by computing the lower approximations or positive region of the decision classes $\{x_1,x_4,x_5\}$ and $\{x_2,x_3\}$ for different $A \subseteq \{a_1,a_2\}$:

- From $\text{POS}_{a_1}(\{x_1,x_4,x_5\}) = \emptyset$ we do not derive certain rules.
- From $\text{POS}_{a_2}(\{x_1,x_4,x_5\}) = \{x_1,x_4,x_5\} = m_{a_2}((a_2,\geq,\text{medium}))$ we derive the certain rule

  If $(a_2,\geq,\text{medium})$, then $(d,=,1)$.
- From $\text{POS}_{\{a_1,a_2\}}(\{x_1,x_4,x_5\}) = \{x_1,x_4,x_5\}$ we do not derive certain rules, as we cannot write $\{x_1,x_4,x_5\}$ as the intersection of the meaning sets of an atomic formula $(a_1,\in,z_1)$ with $z_1 \in L_{a_1}$ and an atomic formula $(a_2,\geq,z_2)$ with $z_2 \in L_{a_2}$.
- From $\text{POS}_A(\{x_2,x_3\}) = \emptyset$ for $A \subseteq \{a_1,a_2\}$ we do not derive certain rules.

3.4 Application: dominance-based rough set models

To illustrate the semantical approach of covering-based rough sets, we determine the elementary and definable sets in the framework of dominance-based rough sets, introduced by Greco, Matarazzo and Słowiński [53, 54, 149]. The model extends the rough set model of Pawlak by using a dominance relation instead of an equivalence relation as indiscernibility relation. It is a good idea to choose a dominance relation instead of an equivalence relation when the domains V_a of the attributes in At are preference-ordered, i.e., if there is a natural order on the possible values of an attribute. A real-life example is the overall evaluation of bank clients based on the evaluations of different risk factors.

Formally, an outranking relation $\succeq_a$ is defined for each attribute $a \in At$ based on the natural order on V_a, i.e., an object $x \in U$ dominates an object $y \in U$, or y is dominated by x, with respect to the attribute a if $I_a(x) \succeq_a I_a(y)$. Such a relation $\succeq_a$ is reflexive and transitive. It is assumed that each relation $\succeq_a$ is complete, i.e., that for every pair of objects one object is dominating the other. This way, we also get preference-ordered decision classes D_i, with $D_i = \{x \in U \mid I_d(x) = i\}$, $i \in V_d$. For $i, j \in V_d$, if $i \succeq_d j$, the objects from D_i are strictly preferred to the objects from D_j. For example, the bank clients with overall evaluation 'good' are preferable to the clients with overall evaluation 'medium'.

As the decision classes are preference-ordered, we obtain the upward and downward union of classes: for $i \in V_d$ we have

$$D_i^{\succeq} = \bigcup \{D_j \in U/d \mid j \succeq_d i\}$$

and

$$D_i^{\preceq} = \bigcup \{D_j \in U/d \mid i \succeq_d j\}.$$

An object x belongs to $D_i^{\succeq}$ if the decision of x is at least i, while x belongs to $D_i^{\preceq}$ if the the decision of x is at most i.

Given a set of conditional attributes $A \subseteq C$, we obtain a relation D_A of U based on A as follows: an object $x \in U$ dominates an object $y \in U$ or y is dominated by x with respect to A if and only if $x \succeq_a y$ for all $a \in A$. The relation D_A is a complete pre-order, since all relations $\succeq_a$ are. Given the relation D_A for $A \subseteq C$ and an object $x \in U$, we can define the A-dominating and A-dominated set of x. The former is given by all predecessors of x by D_A (foreset), the latter by the successors of x (afterset):

$$D_A^p(x) = \{y \in U \mid (y, x) \in D_A\}, \tag{3.1}$$

$$D_A^s(x) = \{y \in U \mid (x, y) \in D_A\}. \tag{3.2}$$

An object y belongs to $D_A^p(x)$ if for all attributes $a \in A$ y dominates x with respect to the attribute a, while y belongs to $D_A^s(x)$ if for all attributes $a \in A$ y is dominated by x with respect to the attribute a. Note that both $D_A^p(x)$ and $D_A^s(x)$ are reflexive and transitive neighborhoods of the object x [182].

To obtain useful knowledge from the decision table, we want to derive decision rules from the given data. More specifically, Greco et al. obtained certain rules from the following lower approximations of the upward and downward union of classes:

$$\underline{\text{apr}}_{D_A^p}(D_i^{\succeq}) = \{x \in U \mid D_A^p(x) \subseteq D_i^{\succeq}\}, \tag{3.3}$$

$$\underline{\text{apr}}_{D_A^s}(D_i^{\preceq}) = \{x \in U \mid D_A^s(x) \subseteq D_i^{\preceq}\}. \tag{3.4}$$

Moreover, the sets $\mathbb{C}_A^p = \{D_A^p(x) \mid x \in U\}$ and $\mathbb{C}_A^s = \{D_A^s(x) \mid x \in U\}$ are coverings of the universe U. These coverings are meaningful families of basic granules for $A \subseteq C$ as it is clear that every patch $D_A^p(x)$, respectively $D_A^s(x)$, represents the objects which attributes values on A are bounded from below, respectively from above, by the values of x on A.

In addition, the definable sets are given by the union-closed sets $S_{\cup,\mathbb{C}_A^p}$ and $S_{\cup,\mathbb{C}_A^s}$. From Section 3.3, the lower approximation of $X \subseteq U$ using $S_{\cup,\mathbb{C}_A^p}$ and $S_{\cup,\mathbb{C}_A^s}$ is given, respectively, by

$$\underline{\text{apr}}_{S_{\cup,\mathbb{C}_A^p}}(X) = \bigcup\{Y \in S_{\cup,\mathbb{C}_A^p} \mid Y \subseteq X\}, \tag{3.5}$$

$$\underline{\text{apr}}_{S_{\cup,\mathbb{C}_A^s}}(X) = \bigcup\{Y \in S_{\cup,\mathbb{C}_A^s} \mid Y \subseteq X\}. \tag{3.6}$$

By the following proposition we see that the lower approximations presented in Eqs. (3.3) and (3.4) are equal to the ones presented in Eqs. (3.5) and (3.6) when we consider the sets $D_i^{\succeq}$ and $D_i^{\preceq}$ respectively. Hence, the computational approximation operators used by Greco et al. both have a semantically sound counterpart, provided by the framework from Section 3.3.

Proposition 3.4.1. [140, 182] Let N be a neighborhood operator and

$$\mathbb{C}_{\mathrm{N}} = \{\mathrm{N}(x) \mid x \in U\}.$$

The operators $\underline{\text{apr}}_{\mathrm{N}}$ and $\underline{\text{apr}}_{S_{\cup,\mathbb{C}_{\mathrm{N}}}}$ are equal if and only if N is reflexive and transitive.

The interpretation of the lower approximation of an upward union $D_i^{\succeq}$ is the following: an object x certainly belongs to $D_i^{\succeq}$, i.e., it belongs to its lower approximation, if for every object y which dominates x with respect to A it holds that the decision of y is at least i. Analogously, x certainly belongs to $D_i^{\preceq}$ if every object y which is dominated by x with respect to A has a decision at most i. This way, if the evaluation of an object on A improves, the class assignment of the object does not worsen and vice versa, if the evaluation on A is less good, the class assignment does not improve. Therefore, it is less meaningful to consider $\{x \in U \mid D_A^p(x) \subseteq D_i^{\preceq}\}$ and $\{x \in U \mid D_A^s(x) \subseteq D_i^{\succeq}\}$ as lower approximations since the interpretation is more complicated, although it can be done from computational point of view.

To obtain certain decision rules, let $A = \{a_1, a_2, \ldots, a_n\} \subseteq C$ and $i \in V_d$. If the lower approximation $\underline{\text{apr}}_{D_A^p}(D_i^{\succeq})$ is not empty then we derive the certain decision rule

$$\text{if } I_{a_1}(x) \succeq_{a_1} v_1 \wedge I_{a_2}(x) \succeq_{a_2} v_2 \wedge \ldots \wedge I_{a_n}(x) \succeq_{a_n} v_n, \text{ then } I_d(x) \succeq_d i,$$

with $v_i \in V_{a_i}$. Analogously, if $\underline{\text{apr}}_{D_A^s}(D_i^{\preceq})$ is not empty, then the following certain decision rule is obtained:

$$\text{if } v_1 \succeq_{a_1} I_{a_1}(x) \wedge v_2 \succeq_{a_2} I_{a_2}(x) \wedge \ldots \wedge v_n \succeq_{a_n} I_{a_n}(x), \text{ then } i \succeq_d I_d(x).$$

In the above discussion, we obtained certain decision rules as we used the lower approximations of the upward and downward unions of decision classes. However, Greco et al. also derived possible rules by using the upper approximations, which we obtain as the dual operators from Eqs. (3.3) and (3.4):

$$\overline{\text{apr}}_{D_A^p}(D_i^{\succeq}) = \{x \in U \mid D_A^s(x) \cap D_i^{\succeq} \neq \emptyset\}, \tag{3.7}$$

$$\overline{\text{apr}}_{D_A^s}(D_i^{\preceq}) = \{x \in U \mid D_A^p(x) \cap D_i^{\preceq} \neq \emptyset\}. \tag{3.8}$$

Note that these upper approximations are obtained from a computational viewpoint, and not from a conceptual one. Although they provide us with possible rules which can be used in data analysis, the semantical meaning of these rules is less clear.

To end, we summarize the different steps to obtain the lower approximation operator in the semantically sound approach and the dominance-based rough set approach in Figure 3.2. The lower approximation operators from both frameworks are equal, but there is no such comparison for the upper approximation operators. By constructing the meaningful coverings $\mathbb{C}^p$ and $\mathbb{C}^s$ via the dominance relation $\succeq$, the dominance-based rough set model can be seen as a special case of the semantically sound framework of rough sets.

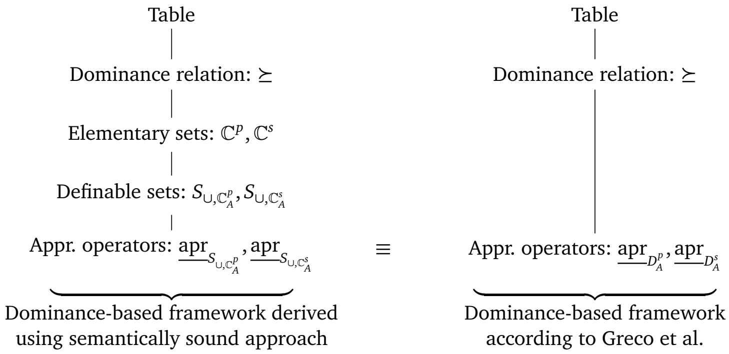


Figure 3.2: Comparison between the semantical framework and the dominance-based rough set model

3.5 Pawlak's rough set model for decision tables with missing values

Given a decision table

$$T = (U, At = C \cup \{d\}, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

then the table is called *incomplete* if at least one information function I_a is incomplete. Note that we will assume that the information function I_d related with the decision attribute d is always complete.

There are two types of incomplete information tables [90]: set-valued information tables [65, 125] which contain information functions

$$I_a : U \to \mathscr{P}(V_a),$$

i.e., a set of attribute values is associated with an object of U, and information tables with missing values [77, 88, 89, 121]. We will concentrate on the latter type.

Furthermore, there are two main strategies to deal with missing attribute values. The first strategy is based on conversion of incomplete data sets into complete data sets, e.g., by replacing missing values by the most common attribute value, or by ignoring cases with missing attribute values [61, 89, 96, 121]. The second strategy acquires the knowledge from the incomplete decision table without preprocessing the data [59].

Recently, Hu and Yao [68] discussed the concept of definability in set-valued incomplete decision tables using an associated family of complete decision tables. However, in this section we discuss definability for incomplete decision tables with missing values without converting the incomplete decision table to a family of complete decision tables. We base ourselves on some papers of Grzymala-Busse [57,59] in which a characteristic set associated with $x \in U$ and $A \subseteq At$ is constructed as the smallest set of instances indiscernible from x by A. However, we will construct a formal language and satisfiability relation in order to define the elementary and definable sets related with a incomplete decision table T.

Note that we discuss a semantical approach for Pawlak's rough set model for incomplete decision tables. Thus, for all the attributes in At only the equality relation $=$ is considered.

We assume that there are three reasons for an attribute value $I_a(x)$ to be missing, for $x \in U$, $a \in C$:

1. The attribute value is a *lost value*. We will denote this by $I_a(x) = ?$. When the attribute value $I_a(x)$ is lost, then the attribute value was originally known, however, it is not currently available. We therefore assume that the original value $I_a(x)$ is one of the attribute values $v \in V_a$.
2. The attribute value is a *do-not-care value*. We will denote this by $I_a(x) = *$. This means that the attribute value $I_a(x)$ is not relevant and may be any value in the domain V_a of the attribute a.
3. The attribute value is an *attribute-concept value*. We will denote this by $I_a(x) = -$. It means that the attribute value $I_a(x)$ is not known, but belongs to the set of typical attribute values $V_{a,x} \subseteq V_a$. The subset of attribute values $V_{a,x}$ is defined by

$$V_{a,x} = \{v \in V_a \mid \exists y \in U : I_d(y) = I_d(x) \text{ with } I_a(y) = v\}, \tag{3.9}$$

i.e., it contains the attribute values of these objects with the same decision as x.

Table 3.3: Incomplete decision table T with $At = \{a_T, a_H, a_N\} \cup \{d\}$

Patient	Temperature a_T	Headache a_H	Nausea a_N	Flu d
x_1	high	$-$	no	yes
x_2	very high	yes	yes	yes
x_3	?	no	no	no
x_4	high	yes	yes	yes
x_5	high	?	yes	no
x_6	normal	yes	no	no
x_7	normal	no	yes	no
x_8	$-$	yes	$*$	yes

We illustrate the concepts of lost value, do-not-care value and attribute-concept value in the following example.

Example 3.5.1. Consider the table T given in Table 3.3 (see [59]). The universe U consists of eight patients and $At = \{a_T, a_H, a_N\} \cup \{d\}$. The conditional attribute a_T records the temperature of the patient and the conditional attributes a_H and a_N state whether the patient has a headache and whether the patient is nauseated. The decision attribute indicates whether the patient has the flu. However, the table T is incomplete. For example, $I_{a_T}(x_3) = ?$ reflects that the temperature of patient x_3 was measured, but e.g., not registered in the computer system. The temperature of patient x_8 was not measured, but as $I_{a_T}(x_8) = -$ and $I_d(x_8) = \text{yes}$, it is assumed that the temperature of patient x_8 lies in the set

$$V_{a_T, x_8} = \{I_{a_T}(y) \in V_{a_T} \mid I_d(y) = I_d(x_8), y \in U\} = \{\text{high}, \text{very high}\}.$$

Moreover, as $I_{a_N}(x_8) = *$, it was considered not relevant for the diagnosis whether patient x_8 was nauseated or not. The missing values $I_{a_H}(x_1)$ and $I_{a_H}(x_5)$ can be interpreted in a similar way.

Incomplete decision tables where all missing attribute values are lost were studied in, i.a., [63, 152]. On the other hand, decision tables were all missing attribute values are do-not-care values were studied in, i.a., [55, 88, 89]. In [58], both lost and do-not-care attribute values are considered. Decision tables in which also attribute-concept values are considered were discussed in [57, 59].

To describe the semantical approach for Pawlak's rough set model in a decision table with missing values, we construct a two-part language and discuss the elementary and definable sets corresponding to this language.

3.5.1 A descriptive language for conjunctive concepts

The descriptive language to describe the conjunctive concepts related with a decision table with missing values is denoted by DLI and is defined as follows:

Definition 3.5.2. The symbols of the *descriptive language* DLI consist of the symbols '=', '(', ')', ',' and '∧', the finite set of attribute symbols $At = C \cup \{d\}$, the finite set of values V_a for each attribute $a \in At$ and the symbols '?, '∗' and '−'. For $a \in C$, let $V_a^+ = V_a \cup \{?, *, -\}$. As we assume I_d to be complete, $V_d^+ = V_d$. The descriptive language DLI is now defined by

1. atomic formulas or atoms $(a, =, v)$ with $a \in At$ and $v \in V_a^+$,
2. if $\varphi, \psi \in$ DLI, then $(\varphi \wedge \psi) \in$ DLI.

Hence, the set of atomic formulas of DLI consists of the atomic formulas of the language DL defined in Section 3.2.1 extended with the atoms $(a, =, ?)$, $(a, =, *)$ and $(a, =, -)$ for each $a \in C$. Moreover, the language DLI is closed under finite conjunction. For $A \subseteq At$, we denote DLI_A for the descriptive language restricted to the attributes of A.

The crucial point in the construction of the definable sets related with DLI_A is the satisfiability relation $\models_A$. We have the following observations [59, 77]:

- If the attribute value $I_a(x)$ is lost, then there is one attribute value $v \in V_a$ such that $I_a(x) = v$, hence, the object x satisfies one of the atomic formulas $(a, =, v)$ with $v \in V_a$.
- If the attribute value $I_a(x)$ is a do-not-care value, then $I_a(x)$ lies in the domain V_a of the attribute a. As it does not matter which attribute value it is, the object x satisfies all atomic formulas $(a, =, v)$ with $v \in V_a$.

- If the attribute value $I_a(x)$ is an attribute-concept value, then we assume $I_a(x) \in V_{a,x}$, hence, the object x satisfies the atomic formulas $(a,=,v)$ with $v \in V_{a,x}$.

Given these interpretations of the different types of missing values, we introduce the satisfiability relation $\models_A$ on $U \times \mathrm{DLI}_A$ as follows:

Definition 3.5.3. Let $x \in U$ and $\varphi \in \mathrm{DLI}_A$ a formula, then the *satisfiability of* φ *by* x *in* DLI_A, denoted by $x \models_A \varphi$, is defined as follows::

1. $x \models_A (a,=,v)$ for $v \in V_a$, $a \in A$, if and only if one of the following cases hold:
   - $I_a(x) = v$,
   - $I_a(x) = *$,
   - $I_a(x) = - \wedge v \in V_{a,x}$,
2. $x \models_A (a,=,?)$ for $a \in A$,
3. $x \models_A (a,=,*)$ for $a \in A$,
4. $x \models_A (a,=,-)$ for $a \in A$ if and only if one of the following conditions hold:
   - $I_a(x) = -$,
   - $I_a(x) = *$,
   - $I_a(x) \in V_a \wedge (\exists y \in U : I_a(y) = - \wedge I_a(x) \in V_{a,y})$,
5. $x \models_A (\varphi \wedge \psi)$ if and only if $x \models_A \varphi$ and $x \models_A \psi$.

We motivate this definition as follows. Let $a \in A$ and $v \in V_a$, then x satisfies $(a,=,v)$ if it is possible that the attribute value $I_a(x)$ equals the value v. This is the case when $I_a(x) = v$. Moreover, it is the case when $I_a(x) = *$, since $I_a(x)$ can be any value in the domain of V_a. In addition, if $I_a(x) = -$, then we assume that $I_a(x) \in V_{a,x}$, hence, $I_a(x)$ can be equal to the value v if and only if $v \in V_{a,x}$.

An object x satisfies the atomic formula $(a,=,?)$ if and only if the attribute value $I_a(x)$ is one of the values in V_a. This holds for all objects $x \in U$. Similarly, every object x satisfies $(a,=,*)$, as $I_a(x)$ lies in the domain V_a of a.

Finally, the atomic formula $(a,=,-)$ is satisfied by the object $x \in U$ if and only if it is possible that $I_a(x)$ belongs to the set of attribute values $V_{a,y}$ for an object $y \in U$ with $I_a(y) = -$. For $I_a(x) = -$, it holds that $I_a(x)$ belongs to $V_{a,x}$, hence, we can choose $y = x$. For $I_a(x) = *$, the value $I_a(x)$ is not relevant, hence, it may be a value in $V_{a,x}$.

We can now define the meaning set of a formula based on the satisfiability relation $\models_A$:

Definition 3.5.4. The *meaning set* $m_A(\varphi)$ of a formula $\varphi \in \mathrm{DLI}_A$ is defined by $m_A(\varphi) = \{x \in U \mid x \models_A \varphi\}$.

As we have discussed before, the meaning set of the conjunction $(\varphi \wedge \psi)$ equals the intersection of the meaning set of φ and the meaning set of ψ. Therefore, it is interesting to study the meaning sets of the atomic formulas. Given the attribute $a \in A$ and $v \in V_a$, then the meaning set of $(a,=,v)$ is given by

$$m_A((a,=,v)) = \{x \in U \mid I_a(x) = v \vee I_a(x) = * \vee (I_a(x) = - \wedge v \in V_{a,x})\}.$$

Moreover, $m_A((a,=,?)) = U$ and $m_A((a,=,*) = U$. In addition, the meaning set of the atomic formula $(a,=,-)$ is given by

$$\begin{aligned} m_A((a,=,-)) = \{x \in U \mid I_a(x) = - \vee I_a(x) = * \vee \\ (I_a(x) \in V_a \wedge (\exists y \in U : I_a(y) = - \wedge I_a(x) \in V_{a,y}))\}. \end{aligned}$$

Note that for $a \in A$ it holds that $m_A((a,=,v)) = m_a((a,=,v))$, for $v \in V_A^+$. Moreover, if the formula φ contains the atomic formula $(a,=,?)$ or $(a,=,*)$, its meaning set is not restricted by these formulas.

Given a formula $\varphi \in \mathrm{DLI}_A$, the meaning set $m_A(\varphi)$ is the extension of the concept with intension φ. A formula is called *valid for the language* DLI_A if its meaning set is the whole universe. Note that for each conditional attribute $a \in A$ the atomic formulas $(a,=,?)$ and $(a,=,*)$ are valid.

Example 3.5.5. By way of illustration, we compute the meaning sets of the atomic formulas of Table 3.3 for $a \in \{a_T, a_H, a_N\}$ in Table 3.4. To this aim, we compute the attribute value sets $V_{a,x}$ for $a \in \{a_T, a_H, a_N\}$ and $x \in U$. For $x \in \{x_1, x_2, x_4, x_8\}$:

$$V_{a_T,x} = \{\text{high}, \text{very high}\},$$

$$\begin{aligned} V_{a_H,x} &= \{\text{yes}\}, \\ V_{a_N,x} &= \{\text{yes}, \text{no}\}. \end{aligned}$$

For $x \in \{x_3, x_5, x_6, x_7\}$ it holds that

$$\begin{aligned} V_{a_T,x} &= \{\text{normal}, \text{high}\}, \\ V_{a_H,x} &= \{\text{yes}, \text{no}\}, \\ V_{a_N,x} &= \{\text{yes}, \text{no}\}. \end{aligned}$$

Again, every definable set is a meaning set of a formula in DLI_A:

Definition 3.5.6. The set of definable sets for the language DLI_A for $A \subseteq At$ is denoted by $\mathrm{DEF}_{\mathrm{DLI}_A}(T)$ and consists of all meaning sets related with DLI_A:

$$\mathrm{DEF}_{\mathrm{DLI}_A}(T) = \{X \subseteq U \mid \exists \varphi \in \mathrm{DLI}_A : X = m_A(\varphi)\}.$$

A set $X \in \mathrm{DEF}_{\mathrm{DLI}_A}(T)$ is called *A-definable*. If a subset $X \subseteq U$ is not A-definable, it is called *A-undefinable* for DLI_A.

Again, the *A-elementary sets* for the language DLI_A form a subset of $\mathrm{DEF}_{\mathrm{DLI}_A}(T)$ and consists of those meaning sets related with a left-hand-side of a rule related with T, i.e., let $A \subseteq C$ and $x \in U$, then the left-hand-side of the rule related with A and x is given by

$$\varphi = \bigwedge_{a \in A} (a, =, I_a(x)).$$

We illustrate the concept of A-elementary sets with the following example.

Example 3.5.7. Given the decision table T presented in Table 3.3, then for $A = C$ and $x_1 \in U$ we have the following formula φ_1:

$$\varphi_1 = (a_T, =, \text{high}) \wedge (a_H, =, -) \wedge (a_N, =, \text{no}).$$

The formula φ_1 describes the left-hand-side of a rule. The meaning set of φ_1 can be computed as follows:

$$m_A(\varphi_1) \quad = \quad m_A((a_T, =, \text{high})) \cap m_A((a_H, =, -)) \cap m_A(a_n, =, \text{no}))$$

Table 3.4: Meaning sets of the atomic formulas of Table 3.3 for $a \in \{a_T, a_H, a_N\}$

Atomic formula	Meaning set
$(a_T, =, \text{normal})$	$\{x_6, x_7\}$
$(a_T, =, \text{high})$	$\{x_1, x_4, x_5, x_8\}$
$(a_T, =, \text{very high})$	$\{x_2, x_8\}$
$(a_T, =, ?)$	U
$(a_T, =, *)$	U
$(a_T, =, -)$	$\{x_1, x_2, x_4, x_5, x_8\}$
$(a_H, =, \text{yes})$	$\{x_1, x_2, x_4, x_6, x_8\}$
$(a_H, =, \text{no})$	$\{x_3, x_7\}$
$(a_H, =, ?)$	U
$(a_H, =, *)$	U
$(a_H, =, -)$	$\{x_1, x_2, x_4, x_6, x_8\}$
$(a_N, =, \text{yes})$	$\{x_2, x_4, x_5, x_7, x_8\}$
$(a_N, =, \text{no})$	$\{x_1, x_3, x_6, x_8\}$
$(a_N, =, ?)$	U
$(a_N, =, *)$	U
$(a_N, =, -)$	$\{x_8\}$

$$= \{x_1, x_4, x_5, x_8\} \cap \{x_1, x_2, x_4, x_6, x_8\} \cap \{x_1, x_3, x_6, x_8\}$$
$$= \{x_1, x_8\},$$

where we have used Table 3.4. We compute the A-elementary sets related with φ_i for $x_i \in U$ in a similar way. The results are presented in Table 3.5.

Table 3.5: A-elementary sets of Table 3.3 for $A = \{a_T, a_H, a_N\}$

Left-hand-side	Meaning set
$\varphi_1 = (a_T, =, \text{high}) \wedge (a_H, =, -) \wedge (a_N, =, \text{no})$	$\{x_1, x_8\}$
$\varphi_2 = (a_T, =, \text{very high}) \wedge (a_H, =, \text{yes}) \wedge (a_N, =, \text{yes})$	$\{x_2, x_8\}$
$\varphi_3 = (a_T, =, ?) \wedge (a_H, =, \text{no}) \wedge (a_N, =, \text{no})$	$\{x_3\}$
$\varphi_4 = (a_T, =, \text{high}) \wedge (a_H, =, \text{yes}) \wedge (a_N, =, \text{yes})$	$\{x_4, x_8\}$
$\varphi_5 = (a_T, =, \text{high}) \wedge (a_H, =, ?) \wedge (a_N, =, \text{yes})$	$\{x_4, x_5, x_8\}$
$\varphi_6 = (a_T, =, \text{normal}) \wedge (a_H, =, \text{yes}) \wedge (a_N, =, \text{no})$	$\{x_6\}$
$\varphi_7 = (a_T, =, \text{normal}) \wedge (a_H, =, \text{no}) \wedge (a_N, =, \text{yes})$	$\{x_7\}$
$\varphi_8 = (a_T, =, -) \wedge (a_H, =, \text{yes}) \wedge (a_N, =, *)$	$\{x_1, x_2, x_4, x_8\}$

As the right-hand-side of rules generally consist of a disjunction of formulas, we extend the descriptive language. This will be done analogously as in Sections 3.2.2 and 3.3.2.

3.5.2 A descriptive language for disjunctions of conjunctive concepts

We extend the descriptive language for a decision table with missing values as follows:

Definition 3.5.8. The *extended descriptive language* EDLI_A for $A \subseteq At$ has the same symbols as the language DLI_A, extended with the symbol '$\vee$'. The formulas of EDLI_A are defined by

1. if $\varphi \in \text{DLI}_A$, then $\varphi \in \text{EDLI}_A$,
2. if $\varphi, \psi \in \text{EDLI}_A$, then $(\varphi \vee \psi) \in \text{EDLI}_A$.

Hence, the descriptive language DLI_A is included in the extended descriptive language EDLI_A. Furthermore, EDLI_A is closed under finite disjunctions.

The satisfiability relation $\vDash_A$ is extended to $U \times \mathrm{EDLI}_A$:

Definition 3.5.9. Let $x \in U$ and $\varphi \in \mathrm{EDLI}_A$ a formula, then the *satisfiability of* φ *by* x *in* EDLI_A, denoted by $x \vDash_A^E \varphi$, is defined in the following way:

1. if $\varphi \in \mathrm{DLI}_A$, then $x \vDash_A^E \varphi$ if and only if $x \vDash_A \varphi$,
2. $x \vDash_A^E (\varphi \vee \psi)$ if and only if $x \vDash_A^E \varphi$ or $x \vDash_A^E \psi$.

We see that the satisfiability relation $\vDash_A^E$ reduces to the relation $\vDash_A$ for formulas in DL_A. Furthermore, an object x satisfies a disjunction of formulas, if it satisfies at least one formula in the disjunction.

The meaning set $m_A^E(\varphi)$ of a formula $\varphi \in \mathrm{EDLI}_A$ represents all the objects which satisfy φ for $\vDash_A^E$:

Definition 3.5.10. Let $\varphi \in \mathrm{EGDL}_A$ a formula, then the *meaning set* $m_A^E(\varphi)$ of φ is given by $m_A^E(\varphi) = \{x \in U \mid x \vDash_A^E \varphi\}$.

Since the satisfiability relation $\vDash_A^E$ reduces to the relation $\vDash_A$ for a formula in DLI_A, the meaning set of formulas of DLI_A does not change for the extended descriptive language EDL_A. Moreover, note that the meaning set of a disjunction of formulas is the union of the meaning sets of the formulas in the disjunction. If $m_A^E(\varphi) = U$, then φ is called *a valid formula in the language* EDLI_A. Note that every formula valid in DLI_A is also valid in EDLI_A.

The meaning sets of EDLI_A provide the definable sets related with this language.

Definition 3.5.11. Given $A \subseteq At$, the A-definable sets of the table T for the extended descriptive language EDLI_A, denoted by $\mathrm{DEF}_{\mathrm{EDLI}_A}(T)$, are the subsets $X \subseteq U$ which are the meaning set of a formula in EDLI_A, i.e., X is *A-definable* if it is the extension of a concept which intension is a formula in EDLI_A. If X is not A-definable, it is called *A-undefinable in* EDLI_A.

We are now able to describe both the left-hand-side and the right-hand-side of a rule, as they are formulas in the language EDLI_{At}. More specifically, the left-hand-side of the rule is a formula in the language DLI_A, with $A \subseteq C$ a set of conditional attributes, and the right-hand-side of the rule is a formula in the language EDLI_d, with d the decision attribute. However, note that we assumed that I_d is complete. Therefore, the language EDLI_d actually reduces to the language EDL_d, which is discussed in Section 3.2.2.

3.5.3 Approximations of undefinable sets

Let $A \subseteq At$ and $X \subseteq U$ an A-undefinable set, then we want to approximate X with A-definable sets from $\text{DEF}_{\text{EDLI}_A}(T)$. We have the following observations for $\text{DEF}_{\text{DLI}_A}(T)$:

- Let $x \in U$ and $a \in A$, then $x \in m_A((a, =, I_a(x))$. Hence, there exists a $X \in \text{DEF}_{\text{DLI}_A}(T)$ such that $x \in X$.
- As illustrated in Table 3.5, the sets in $\text{DEF}_{\text{DLI}_A}(T)$ are not disjoint.

Therefore, $\text{DEF}_{\text{DLI}_A}(T)$ is not a partition, but a covering of the universe U. As the meaning sets of EDLI_A contain those of DLI_A and is closed under union, the set of definable sets $\text{DEF}_{\text{EDLI}_A}(T)$ is the union-closure of the covering $\text{DEF}_{\text{DLI}_A}(T)$. Hence, the approximation operators we obtain for the language EDLI_A are therefore similar to the approximation operators for EGDL_A described in Section 3.3.3:

Definition 3.5.12. Let $X \subseteq U$ and $A \subseteq At$, the *lower and upper approximations of X* for the descriptive language EDLI_A, denoted by $\underline{\text{apr}}_A(X)$ and $\overline{\text{apr}}_A(X)$, are defined by

$$\begin{aligned} \underline{\text{apr}}_A(X) &= \text{ the largest definable set in } \text{EDLI}_A \text{ contained by } X, \\ \overline{\text{apr}}_A(X) &= \{Y \in \text{DEF}_{\text{EDLI}_A}(T) \mid X \subseteq Y, Y \text{ minimal}\}, \end{aligned}$$

with Y minimal if $\forall Z \in \text{DEF}_{\text{EDLI}_A}(T), X \subseteq Z : Z \subseteq Y \Rightarrow Y = Z$.

Hence, the upper approximation of X is not a definable set, but the family of minimal definable sets containing X. Moreover, the positive, negative and boundary region are defined as follows:

$$\begin{aligned}
\mathrm{POS}_A(X) &= \text{the largest definable set in } \mathrm{EDLI}_A \text{ contained by } X,\\
\mathrm{NEG}_A(X) &= \text{the largest definable set in } \mathrm{EDLI}_A \text{ contained by } X^c,\\
\mathrm{BND}_A(X) &= (\mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X))^c.
\end{aligned}$$

3.5.4 Computational approach of Pawlak's rough set model for a decision table with missing values

Similarly as in Section 3.3.4, we obtain that the approximation operators are covering-based approximation operators. For each $a \in At$, we have that

$$\mathbb{C}_a = \{m_a((a, =, v)) \mid m_a((a, =, v)) \neq \emptyset, v \in V_a^+\}.$$

Moreover, for $A \subseteq At$, it holds that

$$\mathbb{C}_A = \left\{K \subseteq U \mid K = \bigcap_{a\in A} K_a, K_a \in \mathbb{C}_a, K \neq \emptyset\right\},$$

i.e., $\mathbb{C}_A$ contains the non-empty intersections $K = \bigcap_{a\in A} K_a$, where we take exactly one set K_a in the intersection from $\mathbb{C}_a$, $a \in A$.

In addition, we obtain that $\mathrm{DEF}_{\mathrm{EDLI}_A}(T) = S_{\cup,\mathbb{C}_A}$. Moreover, we have the following computational characterization for the approximation operators:

$$\begin{aligned}
\underline{\mathrm{apr}}_A(X) &= \bigcup\{K \in S_{\cup,\mathbb{C}_A} \mid K \subseteq X\}\\
&= \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X\},\\
\overline{\mathrm{apr}}_A(X) &= \{K \in S_{\cup,\mathbb{C}_A} \mid X \subseteq K, K \text{ minimal}\},\\
\mathrm{POS}_A(X) &= \bigcup\{K \in S_{\cup,\mathbb{C}_A} \mid K \subseteq X\}\\
&= \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X\},\\
\mathrm{NEG}_A(X) &= \bigcup\{K \in S_{\cup,\mathbb{C}_A} \mid K \subseteq X^c\}\\
&= \bigcup\{K \in \mathbb{C}_A \mid K \subseteq X^c\},\\
\mathrm{BND}_A(X) &= (\mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X))^c\\
&= \bigcap\{K \in \mathbb{C}_A \mid K \cap X \neq \emptyset, K \cap X^c \neq \emptyset\}.
\end{aligned}$$

For $X \subseteq U$ it holds that $\underline{\text{apr}}_A(X)$, $\text{POS}_A(X)$ and $\text{NEG}_A(X)$ are definable sets, $\overline{\text{apr}}_A(X)$ is a set of definable sets and $\text{BND}_A(X)$ is not necessarily a definable set.

Remark 3.5.13. Note that the singleton, subset and concept approximation operators defined in Section 2.2 were introduced for rule induction [56, 62]. However, in [58] it was shown that the singleton approximation operators should not be used for rule induction, due to the fact that the singleton lower approximation operator does not provide a definable set. As the lower subset and lower concept approximation operator are the same, and the upper concept approximation operator yields smaller approximations than the upper subset approximation operator, the author of [57] suggests to use the concept approximation operators for rule induction. It is very easy to see that for $A \subseteq At$ and $X \subseteq U$ the lower concept approximation $\underline{\text{apr}}^A_{\text{conc}}(X)$ coincides with the lower approximation $\underline{\text{apr}}_A(X)$ obtained by the semantical approach for the following neighborhood operator: define the neighborhood operator N^X_A as follows:

$$\forall x \in U : \text{N}^X_A(x) = \bigcup\{K \in \mathbb{C}_A \mid x \in K, K \subseteq X\}. \tag{3.10}$$

To end this section, we illustrate the computational approach for the decision table T presented in Table 3.3 by obtaining certain rules from T.

Example 3.5.14. First, we describe the coverings $\mathbb{C}_A$ for $A \subseteq C$. For $A = \{a_T\}$, $A = \{a_H\}$ and $A = \{a_N\}$, the respective coverings are obtained from Table 3.4:

$$\begin{aligned}
\mathbb{C}_{a_T} &= \{\{x_6,x_7\},\{x_1,x_4,x_5,x_8\},\{x_2,x_8\},U,\{x_1,x_2,x_4,x_5,x_8\}\},\\
\mathbb{C}_{a_H} &= \{\{x_1,x_2,x_4,x_6,x_8\},\{x_3,x_7\},U\},\\
\mathbb{C}_{a_N} &= \{\{x_2,x_4,x_5,x_7,x_8\},\{x_1,x_3,x_6,x_8\},U,\{x_8\}\}.
\end{aligned}$$

Moreover, for $|A| = 2$, we have the following coverings:

$$\begin{aligned}
\mathbb{C}_{\{a_T,a_H\}} &= \{\{x_6\},\{x_7\},\{x_6,x_7\},\{x_1,x_4,x_8\},\{x_1,x_4,x_5,x_8\},\\
&\quad \{x_2,x_8\},\{x_1,x_2,x_4,x_6,x_8\},\{x_3,x_7\},U,\\
&\quad \{x_1,x_2,x_4,x_8\},\{x_1,x_2,x_4,x_5,x_8\}\},\\
\mathbb{C}_{\{a_H,a_N\}} &= \{\{x_2,x_4,x_8\},\{x_1,x_6,x_8\},\{x_1,x_2,x_4,x_6,x_8\},\{x_3\},\{x_7\},\\
&\quad \{x_3,x_7\},U,\{x_2,x_4,x_5,x_7,x_8\},\{x_1,x_3,x_6,x_8\},\{x_8\}\},
\end{aligned}$$

$$\begin{aligned}\mathbb{C}_{\{a_N,a_T\}} &= \{\{x_6\},\{x_7\},\{x_6,x_7\},\{x_4,x_5,x_8\},\{x_1,x_8\},\{x_1,x_4,x_5,x_8\},\\ &\quad \{x_2,x_8\},\{x_8\},\{x_2,x_4,x_5,x_7,x_8\},\{x_1,x_3,x_6,x_8\},U,\\ &\quad \{x_2,x_4,x_5,x_8\},\{x_1,x_2,x_4,x_5,x_8\}\}.\end{aligned}$$

Finally, for $A = C$:

$$\begin{aligned}\mathbb{C}_C &= \{\{x_6\},\{x_7\},\{x_6,x_7\},\{x_4,x_8\},\{x_1,x_8\},\{x_1,x_4,x_8\},\{x_4,x_5,x_8\},\\ &\quad \{x_1,x_4,x_5,x_8\},\{x_2,x_8\},\{x_8\},\{x_2,x_4,x_8\},\{x_1,x_6,x_8\},\{x_3\},U\\ &\quad \{x_1,x_2,x_4,x_6,x_8\},\{x_3,x_7\},\{x_1,x_2,x_4,x_8\},\{x_1,x_3,x_6,x_8\},\\ &\quad \{x_2,x_4,x_5,x_8\},\{x_2,x_4,x_5,x_7,x_8\},\{x_1,x_2,x_4,x_5,x_8\}\}.\end{aligned}$$

Certain rules are obtained by calculating the lower approximation of a decision class. For example, let $A = \{a_H, a_N\}$ and $X = m_d((d, = \text{yes})) = \{x_1, x_2, x_4, x_8\}$, then

$$\begin{aligned}\underline{\text{apr}}_A(X) &= \{x_2,x_4,x_8\}\\ &= \{x_1,x_2,x_4,x_6,x_8\}\cap\{x_2,x_4,x_5,x_7,x_8\}\\ &= m_A((a_H,=,\text{yes}))\cap m_A((a_N,=,\text{yes})).\end{aligned}$$

Hence, we obtain the certain rule

if $(a_H, =, \text{yes})$ and $(a_N, =, \text{yes})$, then $(d, =, \text{yes})$.

Similarly, the following certain rules can be obtained:

- From $\underline{\text{apr}}_{a_T}(\{x_1,x_2,x_4,x_8\}) = \{x_2,x_8\} = m_{a_T}((a_T, =, \text{very high}))$:

  If $(a_T, =, \text{very high})$, then $(d, =, \text{yes})$.

- From $\underline{\text{apr}}_{a_T}(\{x_3,x_5,x_6,x_7\}) = \{x_6,x_7\} = m_{a_T}((a_T, =, \text{normal}))$:

  If $(a_T, =, \text{normal})$, then $(d, =, \text{no})$.

- From $\underline{\text{apr}}_{a_H}(\{x_3,x_5,x_6,x_7\}) = \{x_3,x_7\} = m_{a_H}((a_H, =, \text{no}))$:

  If $(a_H, =, \text{no})$, then $(d, =, \text{no})$.

For no other combination of $A \subseteq C$ and $X \in U/d$ we obtain that $\underline{\text{apr}}_A(X)$ is the intersection of meaning sets, where each meaning set in the intersection relates with exactly one attribute in A.

3.6 Conclusions and future work

In this section, we have discussed three semantically sound approaches to rough set models. First, we revised a semantically sound approach of Pawlak's rough set model for complete decision tables. Here, we constructed a descriptive language in two parts. Moreover, we have described a satisfiability relation and the meaning sets of formulas. The approximation operators are now defined based on the set of definable sets, hence, they are derived concepts of the theory instead of the basic notions to build a rough set model. In addition, we have discussed that the elementary sets related with a set of conditional attributes A is algebraically given by a partition U/E_A, with E_A the regular equivalence relation over the set of attributes A. The set of definable sets is given by the Boolean algebra $\mathscr{B}(U/E_A)$. Furthermore, we have studied how the conceptual approximation operators coincide with the subsystem-based approximation operators of Pawlak's rough set model.

Second, we have extended this semantical approach to complete decision tables with added semantics, allowing for more variety in the choices of relations between the attribute values in the table. The crucial difference between the generalized descriptive language and the descriptive language in the previous approach is the set of atomic formulas, which is a direct consequence of the variation of relations. Therefore, the set of elementary sets is no longer a partition, but a covering $\mathbb{C}_A$. The set of definable sets is given by the union-closure $S_{\cup,\mathbb{C}_A}$ of this covering. Moreover, the upper approximation operator provides not longer a definable set, but it is a set of definable sets. We have studied that the lower approximation operator obtained in this conceptual approach coincides with the tight covering-based lower approximation operator. We have illustrated how this approach may lead to certain rules given a complete decision table with added semantics. In addition, we have applied this semantically sound approach to the theory of dominance-based rough sets in which it is shown that the conceptual lower approximation operator coincides with an element-based approximation operator for a reflexive and transitive neighborhood operator.

Finally, we have introduced a semantically sound approach to decision tables with missing values for Pawlak's rough set model. We have considered three types of missing attribute values: lost values, do-not-care values and attribute-concept values. We have introduced the satisfiability relation for a decision table with these types of missing values. Although only the equality relation is considered as relation between the attribute values, the elementary and definable sets coincide with a covering and its union-closure. As in the previous approach, the conceptual lower approximation operator coincides with the tight covering-based lower approximation operator and the conceptual upper approximation operator provides a set of definable sets instead of a definable set. To end, we have illustrated how this approach provides certain rules related with the table.

The results obtained in Sections 3.3 and 3.5 motivate the research on covering-based rough set models, as it allows rule induction for ordered and incomplete decision tables.

A future research objective is the study of other types of missing values. For example, in [77] it is discussed how an attribute can be *not applicable*: for instance, the attribute 'Pregnant' is not applicable for a male patient. This attribute value should not be considered for this patient.

Another future research objective is the study of set-valued decision tables [97, 127]. If the attribute value of a certain object is a set of values, then this may reflect our incomplete knowledge or it may represents that this object has a few values simultaneously [77]:

- Consider the attribute 'Name': every person can only have one name, hence, if the attribute value for a certain object is a set of names, then we know only one of them is possible. This interpretation is similar to that of a lost attribute value.
- Consider the attribute 'Languages': a person can speak different languages at the same time. Hence, this interpretation is similar to a do-not-care attribute value or an attribute-concept value. The latter is used when there is a limitation on the possible attribute values.

Hence, the study of set-valued decision tables will be very similar to the study of decision tables with missing values.

Some other future research objectives include the following:

- The study of a semantically sound approach to covering-based rough set models when incomplete decision tables are considered, i.e., when we consider a set of relations R_a instead of only the equality relation $=$ for $a \in At$.
- The study of how to obtain meaningful atomic formulas related to a decision table for covering-based rough set models, i.e., how to choose the relations R_a and labels L_a for an attribute $a \in At$.
- The study of rule induction based on upper approximation operators to derive possible rules.

CHAPTER 4

Computational approach of covering-based rough sets

The goal of this chapter is to construct a framework of dual covering-based approximation operators. A first thorough survey of all dual generalizations of Pawlak's model was done by Samanta and Chakraborty in [142, 143], where 16 pairs of approximation operators were considered. The authors discussed properties and implication lattices, in which the implication relations between clusters of inclusions are studied [9]. In 2012, Yao and Yao [189] studied 20 pairs of dual approximation operators. In 2014, Restrepo et al. [141] adopted the framework of Yao and Yao, and also integrated the non-dual framework of Yang and Li [177] into it by considering the corresponding dual lower approximation operators. They reduced the number of different covering approximation operators to 16 dual pairs, and studied the partial order relations between these 16 pairs of dual approximation operators [141], showing which operators yield smaller or larger approximations. Furthermore, in 2016, Zhao [197] studied seven dual pairs of covering-based approximation operators from a topological point of view.

In this chapter, we continue the research on covering-based rough set approximation operators and their partial order relations. We introduce some new approximation operators. In addition, we fuse different frameworks of covering-based approximation operators in order to obtain an overview of different covering-based approximation operators defined in the literature. In Section 4.1, we discuss the equalities and partial order relations between element-based approximation operators based on neighborhood operators defined in [189]. Next, we extend our framework with granule-based and subsystem-based approximation operators in Sections 4.2 and 4.3 and in Section 4.4, with approximation operators related to the framework of Yang and Li [177]. Moreover, we discuss in Sections 4.5 and 4.6 how the frameworks of Zhao [197] and Samanta and Chakraborty [142, 143] correspond with the established framework. Note that some results were already proven in [141]. Finally, we discuss which properties are satisfied by the approximation operators in Section 4.7. Counterexamples for this chapter can be found in Appendix A.

In this chapter and Appendix A, we will often use an abbreviation to denote sets in examples: let $X = \{x_1, x_2, \ldots, x_n\}$ be a set in the universe U, then we will often represent the set X by $x_1 x_2 \ldots x_n$, i.e., we will remove the braces and commas and we write the elements of X as a string.

Example 4.0.1. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{\{1, 2\}, \{3\}, \{1, 2, 3\}\}$ a covering of U, then we will denote $\mathbb{C} = \{12, 3, 123\}$.

Moreover, we consider two partial order relations $\preceq$ and $\leq$ in this chapter. The former is used to describe partial order relations between neighborhood operators and the latter denotes the partial order relation between approximation operators or pairs of dual approximation operators:

- Let N and N′ be two neighborhood operators on U, then we say that $\mathrm{N} \preceq \mathrm{N}'$ if and only if $\forall x \in U : \mathrm{N}(x) \subseteq \mathrm{N}'(x)$.
- Let apr_1 and apr_2 be two approximation operators on U, then we say that $\mathrm{apr}_1 \leq \mathrm{apr}_2$ if and only if $\forall X \subseteq U :\ \mathrm{apr}_1(X) \subseteq \mathrm{apr}_2(X)$.
  Given two pairs of dual approximation operators $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1)$ and $(\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$, we will denote $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1) \leq (\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$ if and only if $\underline{\mathrm{apr}}_1 \leq \underline{\mathrm{apr}}_2$ if and only if $\overline{\mathrm{apr}}_2 \leq \overline{\mathrm{apr}}_1$.

If we use the partial order relation $\leq$ in the latter setting, we indicate that the pair $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ yields a better accuracy than the pair $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$. The *accuracy* $\eta(X)$ of a pair of approximation operators $(\underline{\text{apr}}, \overline{\text{apr}})$ for a set $X \subseteq U$ is defined by the ratio of the cardinalities of the lower approximation of X and the upper approximation of X [128], i.e.,

$$\eta(X) = \frac{|\underline{\text{apr}}(X)|}{|\overline{\text{apr}}(X)|}.$$

We say that a pair $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ is more accurate than a pair $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ if

$$\forall X \subseteq U\colon \frac{|\underline{\text{apr}}_1(X)|}{|\overline{\text{apr}}_1(X)|} \leq \frac{|\underline{\text{apr}}_2(X)|}{|\overline{\text{apr}}_2(X)|}.$$

Since by duality it holds for $X \subseteq U$ that $|\underline{\text{apr}}_1(X)| \leq |\underline{\text{apr}}_2(X)|$ if and only if $|\overline{\text{apr}}_2(X)| \leq |\overline{\text{apr}}_1(X)|$, we derive that pairs with larger lower approximations, and thus smaller upper approximations, provide higher accuracy.

4.1 Element-based approximation operators

Let U be a non-empty universe and $\mathbb{C}$ a covering of U which satisfies the conditions of Proposition 2.2.3. We will assume $\mathbb{C}$ satisfies these conditions in the remainder of this chapter. To establish a unified framework, we discuss the partial order relations between different element-based approximation operators. Therefore, we use the following proposition:

Proposition 4.1.1. Let N and N′ be two neighborhood operators in the covering approximation space $(U, \mathbb{C})$, then $\text{N} \preceq \text{N}'$ if and only if

$$(\underline{\text{apr}}_{\text{N}'}, \overline{\text{apr}}_{\text{N}'}) \leq (\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}}),$$

i.e., $\underline{\text{apr}}_{\text{N}'} \leq \underline{\text{apr}}_{\text{N}}$ and $\overline{\text{apr}}_{\text{N}} \leq \overline{\text{apr}}_{\text{N}'}$.

Proof. Let N and N′ be two neighborhood operators in $(U, \mathbb{C})$. The necessary condition was proven in [141]. For the sufficiency condition, let $x, y \in U$ such that $x \in \text{N}(y)$, then $\text{N}(y) \cap \{x\} \neq \emptyset$ and thus, $y \in \overline{\text{apr}}_{\text{N}}(\{x\})$. Hence, $y \in \overline{\text{apr}}_{\text{N}'}(\{x\})$, thus, $\text{N}'(y) \cap \{x\} \neq \emptyset$. We conclude that $x \in \text{N}'(y)$ and thus, $\text{N} \preceq \text{N}'$. □

Hence, smaller neighborhood operators provide more accurate pairs of dual element-based approximation operators. Moreover, we conclude the following from Proposition 4.1.1:

Corollary 4.1.2. Let N and N′ be two neighborhood operators in the covering approximation space $(U, \mathbb{C})$. Then $\mathrm{N} = \mathrm{N}'$ if and only if

$$(\underline{\mathrm{apr}}_{\mathrm{N}'}, \overline{\mathrm{apr}}_{\mathrm{N}'}) = (\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}}).$$

Therefore, studying equalities and partial order relations between element-based approximation operators reduces to studying equalities and partial order relations between the respective neighborhood operators.

4.1.1 Neighborhood operators based on coverings

In [189], Yao and Yao described four neighborhood operators $N_1^{\mathbb{C}} - N_4^{\mathbb{C}}$ based on $\mathbb{C}$ and five[2] derived coverings $\mathbb{C}_1 - \mathbb{C}_4$ and $\mathbb{C}_\cap$. Hence, combining the four neighborhood operators and six coverings we obtain 24 neighborhood operators $N_i^{\mathbb{C}_j}$, with $i \in \{1,2,3,4\}$ and $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$. For example, let us consider the combination of neighborhood operator N_1 and covering $\mathbb{C}_4$: let $x, y \in U$, then

$$\begin{aligned} y \in N_1^{\mathbb{C}_4}(x) &\Leftrightarrow \forall K \in \mathscr{C}(\mathbb{C}_4, x) \colon y \in K \\ &\Leftrightarrow \forall z \in U \colon x \in \bigcup \mathscr{C}(\mathbb{C}, z) \Rightarrow y \in \bigcup \mathscr{C}(\mathbb{C}, z) \\ &\Leftrightarrow \forall z \in U \colon z \in \bigcup \mathscr{C}(\mathbb{C}, x) \Rightarrow z \in \bigcup \mathscr{C}(\mathbb{C}, y) \\ &\Leftrightarrow \bigcup \mathscr{C}(\mathbb{C}, x) \subseteq \bigcup \mathscr{C}(\mathbb{C}, y) \\ &\Leftrightarrow N_4^{\mathbb{C}}(x) \subseteq N_4^{\mathbb{C}}(y). \end{aligned}$$

In other words, an element y belongs to $N_1^{\mathbb{C}_4}(x)$ if all elements associated with x by $\mathbb{C}$ are also associated with y by $\mathbb{C}$.

In order to study possible equalities between different neighborhood operators, we first show the following results for the minimal and maximal descriptions of some of the derived coverings.

[2]The covering $\mathbb{C}_\cup$ is omitted as it provides the same covering as $\mathbb{C}_1$.

Proposition 4.1.3. Let $(U,\mathbb{C})$ be a covering approximation space and $x \in U$, then

(a) $\mathrm{md}(\mathbb{C}_1,x) = \mathrm{md}(\mathbb{C},x)$,

(b) $\mathrm{md}(\mathbb{C}_2,x) = \mathscr{C}(\mathbb{C}_2,x) = \mathrm{MD}(\mathbb{C}_2,x) = \mathrm{MD}(\mathbb{C},x)$,

(c) $\mathrm{md}(\mathbb{C}_3,x) = \left\{\bigcap \mathrm{md}(\mathbb{C},x)\right\} = \left\{\bigcap \mathscr{C}(\mathbb{C},x)\right\}$,

(d) $\bigcap \mathrm{md}(\mathbb{C}_\cap,x) = \bigcap \mathscr{C}(\mathbb{C}_\cap,x) = \bigcap \mathscr{C}(\mathbb{C},x) = \bigcap \mathrm{md}(\mathbb{C},x)$,

(e) $\mathrm{MD}(\mathbb{C}_\cap,x) = \mathrm{MD}(\mathbb{C},x)$.

Proof. (a) Take $x \in U$, we will prove that $\mathrm{md}(\mathbb{C}_1,x) = \mathrm{md}(\mathbb{C},x)$.

Let $K \in \mathrm{md}(\mathbb{C},x)$, then by definition it holds that $K \in \mathbb{C}_1$ and $x \in K$. We will prove that $K \in \mathrm{md}(\mathbb{C}_1,x)$: let $K' \in \mathbb{C}_1$ with $x \in K'$ and $K' \subseteq K$, then $K' \in \mathbb{C}$ since $\mathbb{C}_1 \subseteq \mathbb{C}$. Because $K \in \mathrm{md}(\mathbb{C},x)$, we have that $K = K'$ and hence, $K \in \mathrm{md}(\mathbb{C}_1,x)$. Therefore, $\mathrm{md}(\mathbb{C},x) \subseteq \mathrm{md}(\mathbb{C}_1,x)$.

On the other hand, let $K \in \mathrm{md}(\mathbb{C}_1,x)$, then $K \in \mathbb{C}_1$ and $x \in K$. Since $\mathbb{C}_1 \subseteq \mathbb{C}$, $K \in \mathbb{C}$ and since $x \in K$, there exists $K' \in \mathrm{md}(\mathbb{C},x)$ with $K' \subseteq K$. Hence, $K' \in \mathbb{C}_1$ and since $K \in \mathrm{md}(\mathbb{C}_1,x)$, we have that $K = K'$. Therefore, $K \in \mathrm{md}(\mathbb{C},x)$ and $\mathrm{md}(\mathbb{C}_1,x) \subseteq \mathrm{md}(\mathbb{C},x)$.

We conclude that $\mathrm{md}(\mathbb{C}_1,x) = \mathrm{md}(\mathbb{C},x)$.

(b)
- Take $x \in U$. We will prove that $\mathrm{md}(\mathbb{C}_2,x) = \mathscr{C}(\mathbb{C}_2,x)$.

  By definition, $\mathrm{md}(\mathbb{C}_2,x) \subseteq \mathscr{C}(\mathbb{C}_2,x)$.

  On the other hand, take $K \in \mathscr{C}(\mathbb{C}_2,x)$ and $K' \in \mathbb{C}_2$ with $x \in K'$ and $K' \subseteq K$. Since $K' \in \mathbb{C}_2$, there exists $y \in U$ such that $K' \in \mathrm{MD}(\mathbb{C},y)$. Since $K' \subseteq K$, $y \in K$ and since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K \in \mathbb{C}$, thus $K \in \mathscr{C}(\mathbb{C},y)$. Since $K' \in \mathrm{MD}(\mathbb{C},y)$ and $K' \subseteq K$, we obtain that $K' = K$. Hence, $K \in \mathrm{md}(\mathbb{C}_2,x)$.

  We conclude that $\mathrm{md}(\mathbb{C}_2,x) = \mathscr{C}(\mathbb{C}_2,x)$.
- Take $x \in U$. We will prove that $\mathscr{C}(\mathbb{C}_2,x) = \mathrm{MD}(\mathbb{C}_2,x)$.

  By definition, $\mathrm{MD}(\mathbb{C}_2,x) \subseteq \mathscr{C}(\mathbb{C}_2,x)$.

  On the other hand, take $K \in \mathscr{C}(\mathbb{C}_2,x)$ and $K' \in \mathbb{C}_2$ with $x \in K'$ and $K \subseteq K'$. Since $K \in \mathbb{C}_2$, there exists $y \in U$ such that $K \in \mathrm{MD}(\mathbb{C},y)$.

Since $K \subseteq K'$, $y \in K'$ and since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K' \in \mathbb{C}$, thus $K' \in \mathscr{C}(\mathbb{C}, y)$. Since $K \in \mathrm{MD}(\mathbb{C}, y)$ and $K \subseteq K'$, we obtain that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}_2, x)$.

We conclude that $\mathscr{C}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}_2, x)$.

– Take $x \in U$. We will prove that $\mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}, x)$.

Let $K \in \mathrm{MD}(\mathbb{C}, x)$, then by definition it holds that $K \in \mathbb{C}_2$ and $x \in K$. We will prove that $K \in \mathrm{MD}(\mathbb{C}_2, x)$: let $K' \in \mathbb{C}_2$ with $x \in K'$ and $K \subseteq K'$, then $K' \in \mathbb{C}$ since $\mathbb{C}_2 \subseteq \mathbb{C}$. Because $K \in \mathrm{MD}(\mathbb{C}, x)$, we have that $K = K'$ and hence, $K \in \mathrm{MD}(\mathbb{C}_2, x)$. Therefore, $\mathrm{MD}(\mathbb{C}, x) \subseteq \mathrm{MD}(\mathbb{C}_2, x)$.

On the other hand, let $K \in \mathrm{MD}(\mathbb{C}_2, x)$, then $K \in \mathbb{C}_2$ and $x \in K$. Since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K \in \mathbb{C}$ and since $x \in K$, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$. Hence, $K' \in \mathbb{C}_2$ and since $K \in \mathrm{MD}(\mathbb{C}_2, x)$, we have that $K = K'$. Therefore, $K \in \mathrm{MD}(\mathbb{C}, x)$ and $\mathrm{MD}(\mathbb{C}_2, x) \subseteq \mathrm{MD}(\mathbb{C}, x)$.

We conclude that $\mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}, x)$.

(c) Take $x \in U$ and $K \in \mathrm{md}(\mathbb{C}_3, x)$. Since $K \in \mathbb{C}_3$, there exists $z \in U$ such that $K = \bigcap \mathscr{C}(\mathbb{C}, z)$. Denote $L = \bigcap \mathscr{C}(\mathbb{C}, x)$, so $L \in \mathbb{C}_3$. We will prove that $K = L$.

Take $y \in L$, then for all $M \in \mathbb{C}$ it holds that if $x \in M$ then $y \in M$. Furthermore, since $x \in K$ it holds for all $M \in \mathbb{C}$ that if $z \in M$ then $x \in M$. Hence, for all $M \in \mathbb{C}$ it holds that if $z \in M$ then $y \in M$, thus, $y \in K$. Hence, $L \subseteq K$ and since $K \in \mathrm{md}(\mathbb{C}_3, x)$ and $L \in \mathbb{C}_3$ with $x \in L$, we conclude that $K = L$.

(d) Take $x \in U$. We will prove that $\bigcap \mathscr{C}(\mathbb{C}_\cap, x) = \bigcap \mathscr{C}(\mathbb{C}, x)$.

Since $\mathbb{C}_\cap \subseteq \mathbb{C}$, we always have $\bigcap \mathscr{C}(\mathbb{C}_\cap, x) \supseteq \bigcap \mathscr{C}(\mathbb{C}, x)$.

For the other inclusion, if $K \in \mathscr{C}(\mathbb{C}, x) \setminus \mathscr{C}(\mathbb{C}_\cap, x)$, then there exists a set $\{K_i \mid i \in I\} \subseteq \mathbb{C}$ such that $K_i \neq K$ for all i and

$$K = \bigcap_{i \in I} K_i.$$

We can assume that $K_i \in \mathbb{C}_\cap$, $\forall i \in I$, otherwise we decompose K_i itself into elements of $\mathbb{C}_\cap$. Moreover, since $x \in K$, $x \in K_i$ for $i \in I$.

Let $y \in \bigcap \mathscr{C}(\mathbb{C}_\cap, x) \setminus \bigcap \mathscr{C}(\mathbb{C}, x)$, then for all $K \in \mathbb{C}_\cap$ it holds that if $x \in K$ then $y \in K$ and there exists $K^* \in \mathbb{C}$ such that $x \in K^*$ and $y \notin K^*$. Hence, there exists $K^* \in \mathscr{C}(\mathbb{C}, x) \setminus \mathscr{C}(\mathbb{C}_\cap, x)$ with $y \notin K^*$. We can decompose K^* into elements of $\mathbb{C}_\cap$ as we saw before: $K^* = \bigcap_{i \in I} K_i$ with $K_i \neq K^*$. Since $y \notin K^*$, there exists K_i with $K_i \in \mathbb{C}_\cap$, $x \in K_i$ and $y \notin K_i$, which is a contradiction.

We conclude that $\bigcap \mathscr{C}(\mathbb{C}_\cap, x) = \bigcap \mathscr{C}(\mathbb{C}, x)$.

(e) Take $x \in U$. We will prove that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$.

First, let us consider $K \in \mathrm{MD}(\mathbb{C}, x)$ and $K' \in \mathbb{C}_\cap$ with $x \in K'$ and $K \subseteq K'$. Since $K' \in \mathbb{C}$ and $K \in \mathrm{MD}(\mathbb{C}, x)$, we have that $K = K'$ and thus $K \in \mathbb{C}_\cap$ and $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$.

On the other hand, take $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$ and $K' \in \mathbb{C}$ with $x \in K'$ and $K \subseteq K'$. If $K' \in \mathbb{C}_\cap$, then $K = K'$. If $K' \notin \mathbb{C}_\cap$, then there exists a set $\{K_i \mid i \in I\} \subseteq \mathbb{C}_\cap \setminus \{K'\}$ with $K' = \bigcap_{i \in I} K_i$. Then for all i, $K \subseteq K_i$ and thus, $K = K_i$ for all i. Again we can conclude that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}, x)$.

We conclude that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$.

□

From Proposition 4.1.3, we derive some more insight into the construction of the different coverings. First, the covering $\mathbb{C}_1$ preserves the minimal description of all the elements. Next, we obtain that the minimal and maximal description in $\mathbb{C}_2$ of an element x correspond to all the sets in $\mathbb{C}_2$ which contain x. Moreover, these sets are exactly the sets of the maximal description in $\mathbb{C}$ of x. Furthermore, we derive that the minimal description of x by $\mathbb{C}_3$ is the singleton $\left\{\bigcap \mathscr{C}(\mathbb{C}, x)\right\}$ which corresponds to the intersection of all elements in $\mathscr{C}(\mathbb{C}, x)$. Next, for $\mathbb{C}_\cap$, we obtain that it preserves the intersection of the sets in $\mathbb{C}$ which contain an arbitrary element x, and this for all elements $x \in U$. This is in line with the idea of $\mathbb{C}_\cap$, i.e., $\mathbb{C}_\cap$ omits intersection reducible sets from $\mathbb{C}$. Moreover, $\mathbb{C}_\cap$ preserves the maximal description in $\mathbb{C}$ of all the elements.

From the results presented in Proposition 4.1.3 we immediately obtain the following equalities between neighborhood operators.

Corollary 4.1.4. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1}$ and $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$,

(b) $N_3^{\mathbb{C}} = N_1^{\mathbb{C}_2} = N_3^{\mathbb{C}_2}$ and $N_4^{\mathbb{C}} = N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}_2}$,

(c) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_3} = N_2^{\mathbb{C}_3}$,

(d) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_\cap}$,

(e) $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_\cap}$ and $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_\cap}$.

The equalities presented in Corollary 4.1.4 are the only equalities between the 24 neighborhood operators $N_i^{\mathbb{C}_j}$. Counterexamples for the other equalities can be found in Counterexamples 1 – 3 in Appendix A and the following two examples.

Example 4.1.5. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{1,12,13\}$, then $N_3^{\mathbb{C}_3}(1) = \{1\}$ and $N_4^{\mathbb{C}_3}(1) = \{1,2,3\}$. Therefore, the neighborhood operators $N_3^{\mathbb{C}_3}$ and $N_4^{\mathbb{C}_3}$ are not equal to each other.

Example 4.1.6. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{12,23,13\}$, then $N_3^{\mathbb{C}}(1) = \{1\}$ and $N_1^{\mathbb{C}_4}(1) = \{1,2,3\}$. Therefore, the neighborhood operators $N_3^{\mathbb{C}}$ and $N_1^{\mathbb{C}_4}$ are not equal to each other.

Hence, the set of 24 neighborhood operators reduces to a set of 13 groups of equal neighborhood operators, presented in Table 4.1.

Besides the neighborhood operators presented in Table 4.1, we can also consider their inverse neighborhood operators defined by

$$\forall x, y \in U : y \in \mathrm{N}^{-1}(x) \Leftrightarrow x \in \mathrm{N}(y).$$

We have the following observations:

Proposition 4.1.7. Let $(U, \mathbb{C})$ be a covering approximation space.

(a) Let N and N′ be two neighborhood operators on $(U, \mathbb{C})$. If $\mathrm{N} \neq \mathrm{N}'$, then $\mathrm{N}^{-1} \neq \mathrm{N}'^{-1}$.

(b) $\left(N_4^{\mathbb{C}}\right)^{-1} = N_4^{\mathbb{C}}$.

Table 4.1: Neighborhood operators $N_i^{\mathbb{C}_j}$ for $(U, \mathbb{C})$

Group	Operators	Group	Operators
$a.$	$N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_2^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}$	$h.$	$N_4^{\mathbb{C}_1}$
$b.$	$N_3^{\mathbb{C}_3}$	$i.$	$N_1^{\mathbb{C}_4}$
$c.$	$N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$	$j.$	$N_4^{\mathbb{C}}, N_2^{\mathbb{C}_2}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
$d.$	$N_3^{\mathbb{C}_1}$	$k.$	$N_2^{\mathbb{C}_4}$
$e.$	$N_2^{\mathbb{C}_\cap}$	$l.$	$N_3^{\mathbb{C}_4}$
$f.$	$N_3^{\mathbb{C}}, N_1^{\mathbb{C}_2}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$	$m.$	$N_4^{\mathbb{C}_4}$
$g.$	$N_4^{\mathbb{C}_3}$		

Proof. (a) Let N and N′ be two neighborhood operators on $(U, \mathbb{C})$ with $\mathrm{N} \neq \mathrm{N}'$, then there exist, without loss of generality, $x, y \in U$ such that $x \in \mathrm{N}(y)$ and $x \notin \mathrm{N}'(y)$. Hence, $y \in \mathrm{N}^{-1}(x)$ and $y \notin \mathrm{N}'^{-1}(y)$, thus $\mathrm{N} \neq \mathrm{N}'$.

(b) Let $x, y \in U$, then

$$\begin{aligned} x \in \left(N_4^{\mathbb{C}}\right)^{-1}(y) &\Leftrightarrow y \in N_4^{\mathbb{C}}(x) \\ &\Leftrightarrow \exists K \in \mathbb{C}: x, y \in K \\ &\Leftrightarrow x \in N_4^{\mathbb{C}}(y). \end{aligned}$$

□

By the first observation, the 13 groups of neighborhood operators provide at most 13 new groups of neighborhood operators and those 13 new groups are all different. Moreover, by the second observation it holds that the neighborhood operator $N_4^{\mathbb{C}}$ is symmetric for every covering $\mathbb{C}$. Therefore, the groups g, h, j and m do not give rise to new neighborhood operators. Hence, there are at most nine new groups. By Counterexamples 1 – 4 from Appendix A and the following example there are no equalities between the first 13 groups and the nine new groups of inverse neighborhood operators.

Example 4.1.8. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{1,12,13\}$, then

(a) $N_3^{\mathbb{C}_3}(2) = N_4^{\mathbb{C}_3}(2) = \{1,2\}$,

(b) $(N_3^{\mathbb{C}_3})^{-1}(2) = \{2\}$.

Therefore, the neighborhood operator $(N_3^{\mathbb{C}_3})^{-1}$ does not equal the operators $N_3^{\mathbb{C}_3}$ or $N_4^{\mathbb{C}_3}$.

Combining the 13 groups of neighborhood operators and the nine groups of inverse neighborhood operators, we derive 22 groups of neighborhood operators in Table 4.2.

We briefly discuss which properties the neighborhood operators satisfy. By definition, it is very easy to see that all neighborhood operators of Table 4.2 are reflexive. Moreover, as discussed above, the neighborhood operators of groups g, h, j and m are symmetric, while the neighborhood operators of the other groups are not. We now want to discuss which neighborhood operators are transitive. To this aim, we have the following proposition:

Proposition 4.1.9. Let $(U,\mathbb{C})$ be a covering approximation space, then

(a) the neighborhood operator $N_1^{\mathbb{C}}$ is transitive,

(b) the neighborhood operator $N_3^{\mathbb{C}}$ is transitive,

(c) if N is a transitive neighborhood operator on U, then N^{-1} is also transitive.

.

Proof. (a) Let $x, y \in U$ and assume $x \in N_1^{\mathbb{C}}(y)$. Now, let $z \in N_1^{\mathbb{C}}(x)$ and $K \in \mathrm{md}(\mathbb{C}, y)$. Since $x \in N_1^{\mathbb{C}}(y)$, $x \in K$. Thus, there exists $K' \in \mathrm{md}(\mathbb{C}, x)$ with $K' \subseteq K$, and $z \in K'$ since $z \in N_1^{\mathbb{C}}(x)$. Hence, $z \in K$ and thus, $z \in N_1^{\mathbb{C}}(y)$.

(b) Let $x, y \in U$ and assume $x \in N_3^{\mathbb{C}}(y)$. Now, let $z \in N_3^{\mathbb{C}}(x)$ and $K \in \mathrm{MD}(\mathbb{C}, y)$. Since $x \in N_3^{\mathbb{C}}(y)$, $x \in K$. Thus, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$, and $z \in K'$ since $z \in N_3^{\mathbb{C}}(x)$. Since $K \in \mathrm{MD}(\mathbb{C}, y)$, $K = K'$. Hence, $z \in K$ and $z \in N_3^{\mathbb{C}}(y)$.

Table 4.2: Neighborhood operators $N_i^{\mathbb{C}_j}$ and their inverse operators for $(U, \mathbb{C})$

Group	Operators	Group	Operators
$a.$	$N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_2^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}$	$a^{-1}.$	$(N_1^{\mathbb{C}})^{-1}, (N_1^{\mathbb{C}_1})^{-1}, (N_1^{\mathbb{C}_3})^{-1}, (N_2^{\mathbb{C}_3})^{-1}, (N_1^{\mathbb{C}_\cap})^{-1}$
$b.$	$N_3^{\mathbb{C}_3}$	$b^{-1}.$	$(N_3^{\mathbb{C}_3})^{-1}$
$c.$	$N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$	$c^{-1}.$	$(N_2^{\mathbb{C}})^{-1}, (N_2^{\mathbb{C}_1})^{-1}$
$d.$	$N_3^{\mathbb{C}_1}$	$d^{-1}.$	$(N_3^{\mathbb{C}_1})^{-1}$
$e.$	$N_2^{\mathbb{C}_\cap}$	$e^{-1}.$	$(N_2^{\mathbb{C}_\cap})^{-1}$
$f.$	$N_3^{\mathbb{C}}, N_1^{\mathbb{C}_2}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$	$f^{-1}.$	$(N_3^{\mathbb{C}})^{-1}, (N_1^{\mathbb{C}_2})^{-1}, (N_3^{\mathbb{C}_2})^{-1}, (N_3^{\mathbb{C}_\cap})^{-1}$
$g.$	$N_4^{\mathbb{C}_3}, (N_4^{\mathbb{C}_3})^{-1}$		
$h.$	$N_4^{\mathbb{C}_1}, (N_4^{\mathbb{C}_1})^{-1}$		
$i.$	$N_1^{\mathbb{C}_4}$	$i^{-1}.$	$(N_1^{\mathbb{C}_4})^{-1}$
$j.$	$N_4^{\mathbb{C}}, N_2^{\mathbb{C}_2}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}, (N_4^{\mathbb{C}})^{-1}, (N_2^{\mathbb{C}_2})^{-1}, (N_4^{\mathbb{C}_2})^{-1}, (N_4^{\mathbb{C}_\cap})^{-1}$		
$k.$	$N_2^{\mathbb{C}_4}$	$k^{-1}.$	$(N_2^{\mathbb{C}_4})^{-1}$
$l.$	$N_3^{\mathbb{C}_4}$	$l^{-1}.$	$(N_3^{\mathbb{C}_4})^{-1}$
$m.$	$N_4^{\mathbb{C}_4}, (N_4^{\mathbb{C}_4})^{-1}$		

(c) Let $x, y, z \in U$ with $x \in \mathrm{N}^{-1}(y)$ and $y \in \mathrm{N}^{-1}(z)$, then $y \in \mathrm{N}(x)$ and $z \in \mathrm{N}(y)$. Hence, by the transitivity of N, it holds that $z \in \mathrm{N}(x)$, and thus $x \in \mathrm{N}^{-1}(z)$. □

We derive that the neighborhood operators in the groups a, a^{-1}, b, b^{-1}, d, d^{-1}, f, f^{-1}, i, i^{-1}, l and l^{-1} are transitive. By considering Counterexample 2 of Appendix A and the following example, we conclude that none of the other neighborhood operators in Table 4.2 are transitive.

Example 4.1.10. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{1, 12, 13\}$, then $1 \in N_4^{\mathbb{C}_3}(2) = \{1,2\}$, but $N_4^{\mathbb{C}_3}(1) = \{1,2,3\} \nsubseteq N_4^{\mathbb{C}_3}(2)$. Hence, $N_4^{\mathbb{C}_3}$ is not transitive.

Next, we study the partial order relations with respect to $\preceq$ for the 22 groups of neighborhood operators.

4.1.2 Partial order relations between neighborhood operators

Each of the 22 groups of neighborhood operators relates to a pair of element-based approximation operators $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$. By Corollary 4.1.4, these pairs are all different. In order to study the partial order relations with respect to $\leq$ between the different pairs of approximation operators, we need to study the partial order relations with respect to $\preceq$ for the different neighborhood operators. First, note that we have the following observation:

Proposition 4.1.11. Let N and N′ be two neighborhood operators on U, then $\mathrm{N} \preceq \mathrm{N}' \Rightarrow \mathrm{N}^{-1} \preceq \mathrm{N}'^{-1}$.

Proof. Let N and N′ such that $\mathrm{N} \preceq \mathrm{N}'$ and assume $x \in \mathrm{N}^{-1}(y)$ for $x, y \in U$. Then $y \in \mathrm{N}(x)$, thus $y \in \mathrm{N}'(x)$ and therefore, $x \in \mathrm{N}'^{-1}(y)$. □

Hence, studying the partial order relations between the groups a – m immediately provides the results for the partial order relations between the groups of inverse neighborhood operators. Also note that as all neighborhood operators from Table 4.2 are different, it holds that if $\mathrm{N} \preceq \mathrm{N}'$, then $\mathrm{N}' \preceq \mathrm{N}$ cannot hold.

We start by fixing the type of neighborhood operator and discuss the partial order relations between the operators of type N_i based on different coverings. We begin with the neighborhood operators of type N_1. Recall that the groups of neighborhood operators which contain an operator of type N_1 are a, f and i.

Proposition 4.1.12. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $N_1^{\mathbb{C}} \preceq N_1^{\mathbb{C}_2}$,

(b) $N_1^{\mathbb{C}_2} \preceq N_1^{\mathbb{C}_4}$.

Proof. (a) We show that $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$ for all $x \in U$. By definition, it holds that $\mathbb{C}_2 \subseteq \mathbb{C}$. Furthermore, for $x \in U$ it holds that $\mathscr{C}(\mathbb{C}_2, x) \subseteq \mathscr{C}(\mathbb{C}, x)$. This implies that $\bigcap \mathscr{C}(\mathbb{C}, x) \subseteq \bigcap \mathscr{C}(\mathbb{C}_2, x)$, so $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$.

(b) Take $x \in U$ and $y \in N_1^{\mathbb{C}_2}(x)$. Then for all $K \in \mathbb{C}_2$ with $x \in K$ it holds that $y \in K$. Take $K' \in \mathscr{C}(\mathbb{C}_4, x)$, then there exists a set $\{K_i \mid i \in I\} \subseteq \mathbb{C}_2$ such that $K' = \bigcup_{i \in I} K_i$. Since $x \in K'$, there exists $i \in I$ such that $x \in K_i$. Hence, $y \in K_i$ and thus $y \in K'$. We conclude that $y \in N_1^{\mathbb{C}_4}(x)$.

□

Hence, in terms of the notation of Table 4.2, we conclude that $a \preceq f \preceq i$.

We continue with neighborhood operators of type N_2. The groups of neighborhood operators which contain an operator of type N_2 are a, c, e, j and k.

Proposition 4.1.13. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $N_2^{\mathbb{C}_3} \preceq N_2^{\mathbb{C}}$,

(b) $N_2^{\mathbb{C}_3} \preceq N_2^{\mathbb{C}_\cap}$,

(c) $N_2^{\mathbb{C}} \preceq N_2^{\mathbb{C}_2}$,

(d) $N_2^{\mathbb{C}_\cap} \preceq N_2^{\mathbb{C}_2}$,

(e) $N_2^{\mathbb{C}_2} \preceq N_2^{\mathbb{C}_4}$.

Proof. (a) Take $x \in U$, then by Proposition 4.1.3(c), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}, x)$. Hence, $N_2^{\mathbb{C}_3}(x) \subseteq \bigcup \mathrm{md}(\mathbb{C}, x) = N_2^{\mathbb{C}}(x)$.

(b) Take $x \in U$, then by Proposition 4.1.3(c), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}, x)$ and by Proposition 4.1.3(d), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}_\cap, x)$. Hence,

$$N_2^{\mathbb{C}_3}(x) \subseteq \bigcup \mathrm{md}(\mathbb{C}_\cap, x) = N_2^{\mathbb{C}_\cap}(x).$$

(c) Take $x \in U$ and $y \in N_2^{\mathbb{C}}(x)$, then there exists $K \in \mathrm{md}(\mathbb{C}, x)$ with $y \in K$. Hence, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$ and thus, $y \in K'$. By Proposition 4.1.3(b), $\mathrm{MD}(\mathbb{C}, x) = \mathrm{md}(\mathbb{C}_2, x)$. Thus, $K' \in \mathrm{md}(\mathbb{C}_2, x)$ and $y \in K'$. Hence, $y \in N_2^{\mathbb{C}_2}(x)$.

(d) Take $x \in U$ and $y \in N_2^{\mathbb{C}_\cap}(x)$, then there exists $K \in \mathrm{md}(\mathbb{C}_\cap, x)$ with $y \in K$. Since $K \in \mathbb{C}_\cap \subseteq \mathbb{C}$, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$ and thus, $y \in K'$. By Proposition 4.1.3(b), $\mathrm{MD}(\mathbb{C}, x) = \mathrm{md}(\mathbb{C}_2, x)$. Thus, $K' \in \mathrm{md}(\mathbb{C}_2, x)$ and $y \in K'$. Hence, $y \in N_2^{\mathbb{C}_2}(x)$.

(e) By Corollary 4.1.4 it holds that $N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}}$. Moreover, we recall that

$$\mathbb{C}_4 = \{N_4^{\mathbb{C}}(x) \mid x \in U\}.$$

Take $x \in U$, we will prove that $N_4^{\mathbb{C}}(x) \subseteq N_2^{\mathbb{C}_4}(x)$.

Denote $\mathrm{md}(\mathbb{C}_4, x) = \{N_4^{\mathbb{C}}(z_i) \mid z_i \in U, i \in I\}$ and assume there is an element $y \in U$ with $y \in N_4^{\mathbb{C}}(x) \setminus N_2^{\mathbb{C}_4}(x)$, i.e., $y \in N_4^{\mathbb{C}}(x)$ and for all $i \in I: y \notin N_4^{\mathbb{C}}(z_i)$. Hence, there exists $K^* \in \mathbb{C}$ with $x, y \in K^*$ and for all $i \in I$ and for all $K \in \mathbb{C}$ it holds that if $y \in K$, then $z_i \notin K$. In other words, $x \in N_4^{\mathbb{C}}(y)$ and $z_i \notin N_4^{\mathbb{C}}(y)$ for all $i \in I$. Since $N_4^{\mathbb{C}}(y) \in \mathbb{C}_4$ and $x \in N_4^{\mathbb{C}}(y)$, there exists $L \in \mathrm{md}(\mathbb{C}_4, x)$ with $L \subseteq N_4^{\mathbb{C}}(y)$. Since $L \in \mathrm{md}(\mathbb{C}_4, x)$, $L = N_4^{\mathbb{C}}(z_i)$ for some $i \in I$, and thus, $N_4^{\mathbb{C}}(z_i) \subseteq N_4^{\mathbb{C}}(y)$. Hence, $z_i \in N_4^{\mathbb{C}}(y)$, which is a contradiction. We conclude that $N_4^{\mathbb{C}}(x) \setminus N_2^{\mathbb{C}_4}(x) = \emptyset$ and thus, $N_4^{\mathbb{C}}(x) \subseteq N_2^{\mathbb{C}_4}(x)$.

□

We can conclude that $a \preceq c \preceq j \preceq k$ and $a \preceq e \preceq j \preceq k$.

Next, we discuss neighborhood operators of type N_3 by considering groups b, d, f and l.

Proposition 4.1.14. Let $(U, \mathbb{C})$ be a covering approximation space, then $N_3^{\mathbb{C}} \preceq N_3^{\mathbb{C}_4}$.

Proof. Take $x \in U$ and $y \in N_3^{\mathbb{C}}(x)$, then for all $K \in \mathrm{MD}(\mathbb{C}, x)$ it holds that $y \in K$. Take $L \in \mathrm{MD}(\mathbb{C}_4, x)$. Since $L \in \mathbb{C}_4$, there exists $z \in U$ such that $L = \bigcup \mathrm{MD}(\mathbb{C}, z)$. Since $x \in L$, there exists $L' \in \mathrm{MD}(\mathbb{C}, z)$ such that $x \in L'$ and thus, there exists $L'' \in \mathrm{MD}(\mathbb{C}, x)$ such that $L' \subseteq L''$. Since $L' \in \mathrm{MD}(\mathbb{C}, z)$, we have that $L' = L''$, thus $L' \in \mathrm{MD}(\mathbb{C}, x)$ and therefore $y \in L'$. We conclude that $y \in L$ and $y \in N_3^{\mathbb{C}_4}(x)$. □

Hence, we conclude that $f \preceq l$.

Finally, we discuss the order relation for neighborhood operators of type N_4. The groups of neighborhood operators which contain an operator of type N_4 are g, h, j and m.

Proposition 4.1.15. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $N_4^{\mathbb{C}_3} \preceq N_4^{\mathbb{C}_1}$,

(b) $N_4^{\mathbb{C}_1} \preceq N_4^{\mathbb{C}}$,

(c) $N_4^{\mathbb{C}} \preceq N_4^{\mathbb{C}_4}$.

Proof. (a) Take $x \in U$ and $y \in N_4^{\mathbb{C}_3}(x)$, then there exists $K \in \mathrm{MD}(\mathbb{C}_3, x)$ with $y \in K$. Then there exists $K' \in \mathbb{C}_1$ with $K \subseteq K'$ and there exists $K'' \in \mathrm{MD}(\mathbb{C}_1, x)$ with $K' \subseteq K''$. Thus, $y \in K'' \subseteq N_4^{\mathbb{C}_1}(x)$.

(b) Take $x \in U$ and $y \in N_4^{\mathbb{C}_1}(x)$, then there exists $K \in \mathrm{MD}(\mathbb{C}_1, x)$ with $y \in K$. Thus, $K \in \mathbb{C}_1 \subseteq \mathbb{C}$, so there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$. Hence, $y \in K'$ and $y \in N_4^{\mathbb{C}}(x)$.

(c) Take $x \in U$ and $y \in N_4^{\mathbb{C}}(x)$, then there exists $K \in \mathrm{MD}(\mathbb{C}, x)$ with $y \in K$. Take $K' = \bigcup \mathrm{MD}(\mathbb{C}, x) \in \mathbb{C}_4$, then $y \in K'$ and there exists $K'' \in \mathrm{MD}(\mathbb{C}_4, x)$ with $K' \subseteq K''$. Hence, $y \in K'' \subseteq N_4^{\mathbb{C}_4}(x)$.

□

In terms of Table 4.2, this means that $g \preceq h \preceq j \preceq m$.

Besides fixing the choice of type of neighborhood operator, it is also possible to fix the covering. For a fixed covering $\mathbb{C}$, the following order relations for neighborhood operators have been established in [141]:

Proposition 4.1.16. [141] Let $(U,\mathbb{C})$ be a covering approximation space, it holds that

(a) $N_1^{\mathbb{C}} \preceq N_2^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$,

(b) $N_1^{\mathbb{C}} \preceq N_3^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$.

These inequalities show that the operators N_1 and N_4 result in the smallest and largest neighborhoods, given a covering $\mathbb{C}$.

Corollary 4.1.17. Let $(U,\mathbb{C})$ be a covering approximation space, then for $\mathbb{C}_1$ we obtain that

- $N_2^{\mathbb{C}_1} \preceq N_4^{\mathbb{C}_1}$,
- $N_1^{\mathbb{C}_1} \preceq N_3^{\mathbb{C}_1} \preceq N_4^{\mathbb{C}_1}$.

Moreover, for $\mathbb{C}_3$ it holds that

- $N_1^{\mathbb{C}_3} \preceq N_3^{\mathbb{C}_3} \preceq N_4^{\mathbb{C}_3}$,

and for $\mathbb{C}_4$ that

- $N_2^{\mathbb{C}_4} \preceq N_4^{\mathbb{C}_4}$,
- $N_1^{\mathbb{C}_4} \preceq N_3^{\mathbb{C}_4} \preceq N_4^{\mathbb{C}_4}$.

Until now, we either fixed the covering, or the type of neighborhood operator. However, we also have the following order relation between $N_1^{\mathbb{C}_4}$ and $N_4^{\mathbb{C}}$.

Proposition 4.1.18. Let $(U,\mathbb{C})$ be a covering approximation space, then $N_1^{\mathbb{C}_4} \preceq N_4^{\mathbb{C}}$.

Proof. For $x \in U$, $N_4^{\mathbb{C}}(x) \in \mathbb{C}_4$ and $x \in N_4^{\mathbb{C}}(x)$. Thus, there exists a $L \in \mathrm{md}(\mathbb{C}_4, x)$ with $L \subseteq N_4^{\mathbb{C}}(x)$. Hence, $N_1^{\mathbb{C}_4}(x) \subseteq L \subseteq N_4^{\mathbb{C}}(x)$. □

We conclude that $i \preceq j$.

The partial order relations discussed above are the only ones between the groups a – m. Counterexamples for the other partial order relations can be found in Counterexamples 1 – 4 of Appendix A and the following three examples.

Example 4.1.19. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{3,12,13,123\}$, then $N_3^{\mathbb{C}_1}(3) = \{1,3\}$ and $N_3^{\mathbb{C}_3}(3) = N_4^{\mathbb{C}_3}(3) = \{3\}$. Therefore, $N_3^{\mathbb{C}_1} \preceq N_3^{\mathbb{C}_3}$ and $N_3^{\mathbb{C}_1} \preceq N_4^{\mathbb{C}_3}$ do not hold.

Example 4.1.20. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1,12,23,24,123,124\}$, then $N_4^{\mathbb{C}_3}(2) = \{2,3,4\}$ and $N_3^{\mathbb{C}_3}(2) = \{2\}$. Therefore, $N_4^{\mathbb{C}_3} \preceq N_3^{\mathbb{C}_3}$ does not hold. Moreover, $N_2^{\mathbb{C}}(2) = \{1,2,3,4\}$ and $N_2^{\mathbb{C}_\cap}(2) = \{2,3,4\}$. Therefore, $N_2^{\mathbb{C}} \preceq N_2^{\mathbb{C}_\cap}$ does not hold.

Example 4.1.21. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{12,23,13\}$, then $N_3^{\mathbb{C}}(1) = \{1\}$ and $N_1^{\mathbb{C}_4}(1) = \{1,2,3\}$. Therefore, $N_1^{\mathbb{C}_4} \preceq N_3^{\mathbb{C}}$ does not hold.

The Hasse diagram with respect to the partial order relation $\preceq$ for the neighborhood operators a – m can be found in Figure 4.1a. A Hasse diagram is a mathematical diagram which represents a finite partially ordered set $(P,\leq)$: each element $p \in P$ is represented by a vertex, and there is a directed edge from the element p to the element q if $p \leq q$ and if there is no $r \in P \setminus \{p,q\}$ with $p \leq r \leq q$. Note that the Hasse diagram presented in Figure 4.1a represents a lattice, with a the minimum and m the maximum.

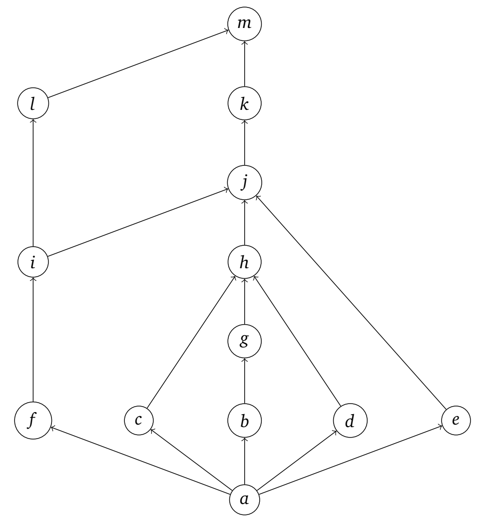
Moreover, by Proposition 4.1.11, we immediately obtain the Hasse diagram with respect to $\preceq$ for the inverse neighborhood operators of a – m in Figure 4.1b. This is also a lattice, with a^{-1} the minimum and $m^{-1} = m$ the maximum.

In addition, we need to study the partial order relations between the 13 groups of neighborhood operators and the nine groups of inverse neighborhood operators. From Figures 4.1a and 4.1b we immediately obtain the following:

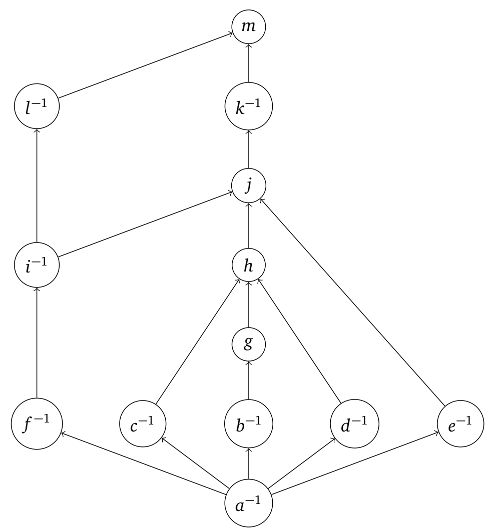
Proposition 4.1.22. Let $(U,\mathbb{C})$ be a covering approximation space.

(a) For $\mathrm{N} \in \{a,b,c,d,e,f,i\}$ it holds that $\mathrm{N} \preceq k^{-1}$.

(b) For $\mathrm{N} \in \{a^{-1},b^{-1},c^{-1},d^{-1},e^{-1},f^{-1},i^{-1}\}$ it holds that $\mathrm{N} \preceq k$.



(a) Hasse diagram of the neighborhood operators $a - m$



(b) Hasse diagram of the inverse neighborhood operators of $a - m$

These are the only partial order relations which hold between the 13 groups of neighborhood operators and the nine groups of inverse neighborhood operators. Counterexamples can be found in Counterexamples 1 – 4 of Appendix A and the following two examples.

Example 4.1.23. Let $U = \{1,2,3,4,5,6\}$ and $\mathbb{C} = \{123, 145, 26\}$, then it holds that $\mathrm{N}(3) = \{1,2,3\}$ for $\mathrm{N} \in \{N_1^{\mathbb{C}}, N_3^{\mathbb{C}_3}, N_3^{\mathbb{C}_1}, N_4^{\mathbb{C}_3}\}$ and $(N_3^{\mathbb{C}_3})^{-1}(3) = \{3\}$. Therefore, $\mathrm{N} \preceq (N_3^{\mathbb{C}_3})^{-1}$ does not hold for $\mathrm{N} \in \{N_1^{\mathbb{C}}, N_3^{\mathbb{C}_3}, N_3^{\mathbb{C}_1}, N_4^{\mathbb{C}_3}\}$. Moreover, for $\mathrm{N} \in \{(N_1^{\mathbb{C}})^{-1}, (N_3^{\mathbb{C}_3})^{-1}, (N_3^{\mathbb{C}_1})^{-1}\}$ it holds that $\mathrm{N}(2) = \{2,3,6\}$ and $N_3^{\mathbb{C}_3}(2) = \{2\}$. Therefore, $\mathrm{N} \preceq N_3^{\mathbb{C}_3}$ does not hold for $\mathrm{N} \in \{(N_1^{\mathbb{C}})^{-1}, (N_3^{\mathbb{C}_3})^{-1}, (N_3^{\mathbb{C}_1})^{-1}\}$.

Example 4.1.24. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{3, 12, 13, 123\}$, then $N_4^{\mathbb{C}_3}(1) = \{1,2\}$ and $(N_3^{\mathbb{C}_1})^{-1}(1) = \{1,2,3\}$. Therefore, $(N_3^{\mathbb{C}_1})^{-1} \preceq N_4^{\mathbb{C}_3}$ does not hold.

The Hasse diagram with respect to $\preceq$ for the 22 neighborhood operators presented in Table 4.2 is given in Figure 4.2. Based on the results obtained in Figure 4.2, we derive partial order relations for the element-based approximation operators related with the neighborhood operators of Table 4.2.

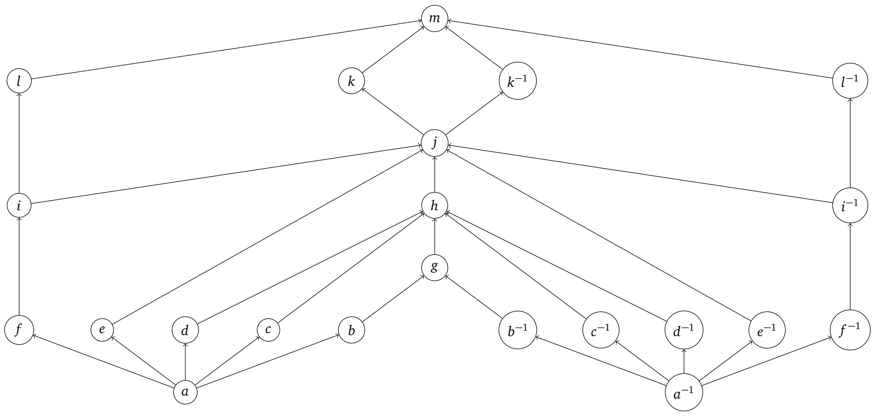


Figure 4.2: Hasse diagram for the neighborhood operators in Table 4.2

4.1.3 Partial order relations between element-based approximation operators

Each of the 22 groups of neighborhood operators from Table 4.2 generates a pair of element-based approximation operators $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$. We list the approximation operators in Table 4.4: the pairs 1 – 22 represent the element-based approximation operators related to groups of Table 4.2. To avoid overload on notation, we use the lowercases $a - m$ and $a^{-1} - l^{-1}$ to denote the neighborhood operators.

The Hasse diagram with respect to the partial order relation $\leq$ for the upper approximation operators of pairs 1 – 22 in Table 4.4 is presented in Figure 4.3. Due to Proposition 4.1.1 all the information we need to construct Figure 4.3 is given in Figure 4.2.

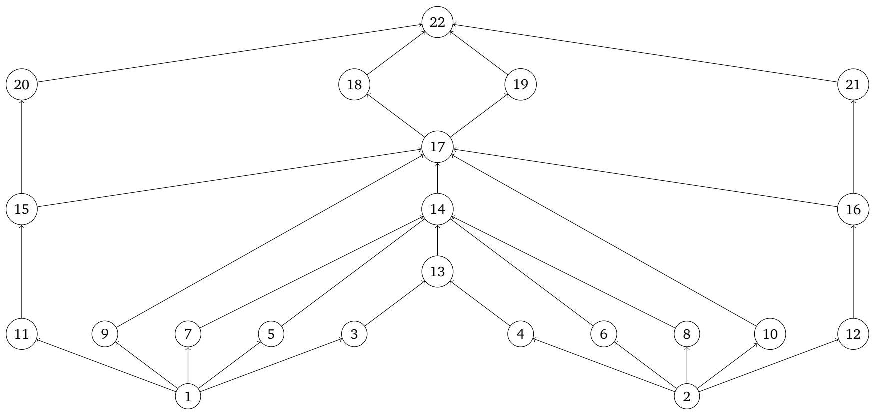


Figure 4.3: Hasse diagram for the upper approximation operators of pairs 1 – 22 in Table 4.4

From Figure 4.3 we derive that the pairs of approximation operators $(\underline{\mathrm{apr}}_{a}, \overline{\mathrm{apr}}_{a})$ and $(\underline{\mathrm{apr}}_{a^{-1}}, \overline{\mathrm{apr}}_{a^{-1}})$ yield the highest accuracies, while the pair of approximation operators $(\underline{\mathrm{apr}}_{m}, \overline{\mathrm{apr}}_{m})$ yields the smallest.

4.2 Granule-based approximation operators

We continue with studying the granule-based approximation operators $(\underline{\text{apr}}'_{\mathbb{C}_j}, \overline{\text{apr}}'_{\mathbb{C}_j})$ and $(\underline{\text{apr}}''_{\mathbb{C}_j}, \overline{\text{apr}}''_{\mathbb{C}_j})$ for $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$.

For the tight approximation operators, the following equalities are proven in [140]:

Proposition 4.2.1. [140] Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $(\underline{\text{apr}}'_{\mathbb{C}_1}, \overline{\text{apr}}'_{\mathbb{C}_1}) = (\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$,

(b) $(\underline{\text{apr}}'_{\mathbb{C}_3}, \overline{\text{apr}}'_{\mathbb{C}_3}) = (\underline{\text{apr}}_{N_1^{\mathbb{C}}}, \overline{\text{apr}}_{N_1^{\mathbb{C}}})$.

None of the granule-based pairs with $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_4, \mathbb{C}_\cap\}$ equals an element-based pair of approximation operators, as they are not a join morphism. In addition, there are no other equalities between pairs of tight approximation operators. Therefore, we can add four new pairs to Table 4.4, numbered 23 to 26.

Moreover, in [140] it is proven that each pair of loose approximation operators is equal to a pair of element-based approximation operators. This result is based on a result presented in [179].

Proposition 4.2.2. [179] Suppose $(\underline{\text{apr}}, \overline{\text{apr}})$ is a dual pair of approximation operators on U. The upper approximation operator $\overline{\text{apr}}$ satisfies $\overline{\text{apr}}(\emptyset) = \emptyset$ and for all $X_i \subseteq U$, $i \in I$, : $\overline{\text{apr}}\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} \overline{\text{apr}}(X_i)$, i.e., $\overline{\text{apr}}$ is a complete join morphism, if and only if there exists a relation R such that

$$(\underline{\text{apr}}, \overline{\text{apr}}) = (\underline{\text{apr}}_R, \overline{\text{apr}}_R),$$

with $\forall x, y \in U \colon (x, y) \in R \Leftrightarrow y \in \overline{\text{apr}}(\{x\})$.

Based on this proposition, we present the following proposition:

Proposition 4.2.3. Let $(U, \mathbb{C})$ a covering approximation space, then

$$(\underline{\text{apr}}''_{\mathbb{C}_j}, \overline{\text{apr}}''_{\mathbb{C}_j}) = (\underline{\text{apr}}_{N_4^{\mathbb{C}_j}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_j}}),$$

for all $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$.

Proof. Let $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$. It is easy to see that $\overline{\text{apr}}''_{\mathbb{C}_j}(\emptyset) = \emptyset$. Additionally, let $\{X_i \subseteq U \mid i \in I\}$ be a family of subsets of the universe U, then

$$\begin{aligned}
\overline{\text{apr}}''_{\mathbb{C}_j}\left(\bigcup_{i\in I} X_i\right) &= \bigcup\{K \in \mathbb{C}_j \mid K \cap \left(\bigcup_{i\in I} X_i\right) \neq \emptyset\} \\
&= \bigcup\{K \in \mathbb{C}_j \mid \bigcup_{i\in I}(K \cap X_i) \neq \emptyset\} \\
&= \bigcup \bigcup_{i\in I}\{K \in \mathbb{C}_j \mid K \cap X_i \neq \emptyset\} \\
&= \bigcup_{i\in I} \bigcup\{K \in \mathbb{C}_j \mid K \cap X_i \neq \emptyset\} \\
&= \bigcup_{i\in I} \overline{\text{apr}}''_{\mathbb{C}_j}(X_i).
\end{aligned}$$

Hence, by Proposition 4.2.2 it holds that $(\underline{\text{apr}}''_{\mathbb{C}_j}, \overline{\text{apr}}''_{\mathbb{C}_j}) = (\underline{\text{apr}}_R, \overline{\text{apr}}_R)$, with R defined by, for $x, y \in U$:

$$\begin{aligned}
(x, y) \in R &\Leftrightarrow y \in \overline{\text{apr}}''_{\mathbb{C}_j}(\{x\}) \\
&\Leftrightarrow y \in \bigcup\{K \in \mathbb{C}_j \mid K \cap \{x\} \neq \emptyset\} \\
&\Leftrightarrow y \in \bigcup\{K \in \mathbb{C}_j \mid x \in K\} \\
&\Leftrightarrow y \in N_4^{\mathbb{C}_j}(x).
\end{aligned}$$

Therefore, we conclude that $(\underline{\text{apr}}''_{\mathbb{C}_j}, \overline{\text{apr}}''_{\mathbb{C}_j}) = (\underline{\text{apr}}_{N_4^{\mathbb{C}_j}}, \overline{\text{apr}}_{N_4^{\mathbb{C}_j}})$. □

Thus, the tight approximation operators yield no new pairs in Table 4.4.

To study the partial order relations with respect to $\leq$, we need to study for each pair $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ of the pairs 23 – 26 and for each pair $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ of the pairs 1 – 26 in Table 4.4 whether one pair has a larger accuracy than the other or whether both pairs are incomparable with respect to $\leq$. .

In [141], the following partial order relations between upper approximation operators were proven:

Proposition 4.2.4. [141] Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $\overline{\text{apr}}'_{\mathbb{C}_3} \leq \overline{\text{apr}}'_{\mathbb{C}_1} \leq \overline{\text{apr}}'_{\mathbb{C}_\cap} \leq \overline{\text{apr}}'_{\mathbb{C}_2} \leq \overline{\text{apr}}'_{\mathbb{C}_4}$,

(b) $\overline{\text{apr}}'_{\mathbb{C}} \leq \overline{\text{apr}}_{N_2^{\mathbb{C}}}$,

(c) $\overline{\text{apr}}_{N_3^{\mathbb{C}}} \leq \overline{\text{apr}}'_{\mathbb{C}_2}$,

(d) $\overline{\text{apr}}'_{\mathbb{C}_4} \leq \overline{\text{apr}}_{N_4^{\mathbb{C}}}$.

Furthermore, we have the following two propositions:

Proposition 4.2.5. Let $(U, \mathbb{C})$ be a covering approximation space, then it holds that $\overline{\text{apr}}_{N_1^{\mathbb{C}_4}} \leq \overline{\text{apr}}'_{\mathbb{C}_4}$.

Proof. Let $X \subseteq U$ and $x \in \overline{\text{apr}}_{N_1^{\mathbb{C}_4}}(X)$, then $N_1^{\mathbb{C}_4}(x) \cap X \neq \emptyset$. Now, for all $K \in \mathbb{C}_4$ with $x \in K$ it holds that $N_1^{\mathbb{C}_4}(x) \subseteq K$, hence, $K \cap X \neq \emptyset$. We conclude $x \in \overline{\text{apr}}'_{\mathbb{C}_4}(X)$. □

Proposition 4.2.6. Let $(U, \mathbb{C})$ be a covering approximation space, then it holds that $\overline{\text{apr}}'_{\mathbb{C}_\cap} \leq \overline{\text{apr}}_{N_2^{\mathbb{C}_\cap}}$.

Proof. Let $X \subseteq U$ and $x \in U$ such that $x \in \overline{\text{apr}}'_{\mathbb{C}_\cap}(X)$, then for all $K \in \mathbb{C}_\cap$ it holds that $x \in K$ implies $K \cap X \neq \emptyset$. Hence, there exists $K \in \text{md}(\mathbb{C}_\cap, x)$ it holds that that $K \cap X \neq \emptyset$, thus $N_2^{\mathbb{C}_\cap}(x) \cap X \neq \emptyset$. We conclude $x \in \overline{\text{apr}}_{N_2^{\mathbb{C}_\cap}}(X)$. □

Figure 4.4 represents the above partial order relations, as well as all the remaining partial order relations which hold by transitivity of ≤ for pairs 1 – 26 of Table 4.4. Counterexamples for the other partial order relations can be found in Counterexample 5 of Appendix A and the following three examples.

Example 4.2.7. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{12, 13, 24, 34\}$, then it holds that $\overline{\text{apr}}_1(\{1, 4\}) = \{1, 4\}$ and $\overline{\text{apr}}_2(\{1, 4\}) = \{1, 2, 3, 4\}$, hence $\overline{\text{apr}}_2 \leq \overline{\text{apr}}_1$ does not hold for

$$\overline{\text{apr}}_1 \in \{\overline{\text{apr}}_b, \overline{\text{apr}}_{b^{-1}}, \overline{\text{apr}}_g, \overline{\text{apr}}_i, \overline{\text{apr}}_{i^{-1}}, \overline{\text{apr}}_l, \overline{\text{apr}}_{l^{-1}}\}$$

and

$$\overline{\text{apr}}_2 \in \{\overline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}_\cap}, \overline{\text{apr}}'_{\mathbb{C}_2}, \overline{\text{apr}}'_{\mathbb{C}_4}\}.$$

Example 4.2.8. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{1, 12, 23, 24, 123, 234\}$, then it holds that $\overline{\text{apr}}_h(\{1, 4\}) = \{1, 2, 4\}$ and $\overline{\text{apr}}'_{\mathbb{C}_\cap}(\{1, 4\}) = \{1, 2, 3, 4\}$. Hence, $\overline{\text{apr}}'_{\mathbb{C}_\cap} \leq \overline{\text{apr}}_h$ does not hold. Moreover,

(a) $\overline{\text{apr}}_e(\{2\}) = \{2, 3, 4\}$,

(b) $\overline{\text{apr}}_{e^{-1}}(\{2\}) = \{1,2,4\}$,

(c) $\overline{\text{apr}}'_{\mathbb{C}}(\{2\}) = \overline{\text{apr}}'_{\mathbb{C}_\cap}(\{2\}) = \{2,3,4\}$,

(d) $\overline{\text{apr}}'_{\mathbb{C}_2}(\{2\}) = \{1,2,3,4\}$.

Hence, $\overline{\text{apr}}'_{\mathbb{C}} \leq \overline{\text{apr}}_{e^{-1}}$, $\overline{\text{apr}}'_{\mathbb{C}_\cap} \leq \overline{\text{apr}}_{e^{-1}}$, $\overline{\text{apr}}'_{\mathbb{C}_2} \leq \overline{\text{apr}}_{e^{-1}}$ and $\overline{\text{apr}}'_{\mathbb{C}_2} \leq \overline{\text{apr}}_{e}$ do not hold.

Example 4.2.9. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1,2,12,23,14\}$, then it holds that $\overline{\text{apr}}_{c^{-1}}(\{2\}) = \{2\}$ and $\overline{\text{apr}}'_{\mathbb{C}}(\{2\}) = \overline{\text{apr}}'_{\mathbb{C}_\cap}(\{2\}) = \{2,3\}$, hence, $\overline{\text{apr}}'_{\mathbb{C}} \leq \overline{\text{apr}}_{c^{-1}}$ and $\overline{\text{apr}}'_{\mathbb{C}_\cap} \leq \overline{\text{apr}}_{c^{-1}}$ do not hold. Moreover, $\overline{\text{apr}}_d(\{3\}) = \{2,3\}$ and $\overline{\text{apr}}'_{\mathbb{C}_4}(\{3\}) = \{3\}$, hence, $\overline{\text{apr}}_d \leq \overline{\text{apr}}'_{\mathbb{C}_4}$ does not hold.

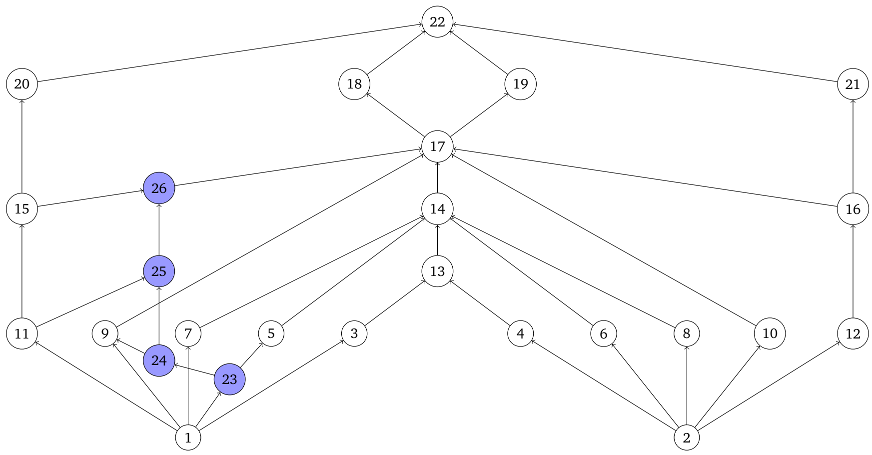


Figure 4.4: Hasse diagram for the upper approximation operators of pairs 1 – 26 in Table 4.4, where we have extended the Hasse diagram presented in Figure 4.3 with pairs 23 – 26.

4.3 Subsystem-based approximation operators

Next, we want to add the two pairs of subsystem-based approximation operators $(\underline{\text{apr}}_{S_\cap}, \overline{\text{apr}}_{S_\cap})$ and $(\underline{\text{apr}}_{S_\cup}, \overline{\text{apr}}_{S_\cup})$ to Figure 4.4. In [140], the following equality was proven:

Proposition 4.3.1. [140] Let $(U, \mathbb{C})$ be a covering approximation space, then $(\underline{\text{apr}}_{S_\cup}, \overline{\text{apr}}_{S_\cup}) = (\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$.

The pair $(\underline{\text{apr}}_{S_\cap}, \overline{\text{apr}}_{S_\cap})$ defines a pair of approximation operators different from the pairs 1 – 26 presented in Table 4.4. There is only one partial order relation which holds:

Proposition 4.3.2. [141] Let $(U, \mathbb{C})$ be a covering approximation space, then $\overline{\text{apr}}_{a^{-1}} \leq \overline{\text{apr}}_{S_\cap}$.

To see that no upper approximation operator of pairs 1 – 26 from Table 4.4 yields a larger approximation than $\overline{\text{apr}}_{S_\cap}$, consider the following example.

Example 4.3.3. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1,2,3,4\}$, then $\overline{\text{apr}}_m(\{1,2\}) = \{1,2\}$ and $\overline{\text{apr}}_{S_\cap}(\{1,2\}) = \{1,2,3,4\}$. As $\overline{\text{apr}}_{S_\cap}$ does not provide a smaller upper approximation than $\overline{\text{apr}}_m$, it does not provide a smaller upper approximation than any of the upper approximation operators of pairs 1 – 26, hence, $\overline{\text{apr}}_{S_\cap} \leq \overline{\text{apr}}$ does not hold for any $\overline{\text{apr}}$ of pairs 1 – 26 in Table 4.4.

Moreover, note from the previous example that when the considered covering is a partition, the pair $(\underline{\text{apr}}_{S_\cap}, \overline{\text{apr}}_{S_\cap})$ does not necessarily coincide with Pawlak's rough set approximation operators. To see that no other upper approximation operator other than $\overline{\text{apr}}_{a^{-1}}$ is smaller than $\overline{\text{apr}}_{S_\cap}$, consider Counterexample 5 in Appendix A.

We represent the new Hasse diagram with respect to $\leq$ for pairs 1 – 27 of Table 4.4 in Figure 4.5.

4.4 Framework of Yang and Li

In [177], Yang and Li studied seven upper approximation operators, which we consider with their dual lower approximation operator. As previously discussed in Chapter 2, it is clear that $H_2^{\mathbb{C}} = \overline{\text{apr}}''_{\mathbb{C}}$ and $H_6^{\mathbb{C}} = \overline{\text{apr}}_{N_1^{\mathbb{C}}}$. Moreover, in [140] the following equality is proven:

Proposition 4.4.1. [140] Let $(U, \mathbb{C})$ be a covering approximation space, then $H_7^{\mathbb{C}} = \overline{\text{apr}}''_{\mathbb{C}_3}$.

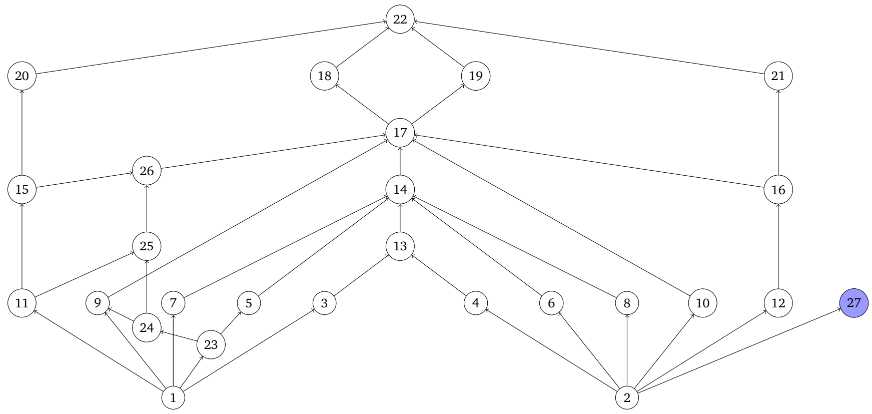


Figure 4.5: Hasse diagram for the upper approximation operators of pairs 1 – 27 in Table 4.4, where we have extended the Hasse diagram presented in Figure 4.4 with pair 27.

Moreover, given a neighborhood operator N, we can prove the following equality:

Proposition 4.4.2. Let$(U, \mathbb{C})$ be a covering approximation space and N a neighborhood operator, then $\overline{\text{apr}}_{\text{conc}} = \overline{\text{apr}}_{\text{N}^{-1}}$.

Proof. Let $X \subseteq U$ and $x \in U$, then

$$\begin{aligned} x \in \overline{\text{apr}}_{\text{conc}}(X) &\Leftrightarrow \exists y \in X : x \in \text{N}(y) \\ &\Leftrightarrow \exists y \in X : y \in \text{N}^{-1}(x) \\ &\Leftrightarrow \text{N}^{-1}(x) \cap X \neq \emptyset \\ &\Leftrightarrow x \in \overline{\text{apr}}_{\text{N}^{-1}}(X). \end{aligned}$$

□

Therefore, the following equalities hold for $H_3^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$.

Corollary 4.4.3. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $H_3^{\mathbb{C}} = \overline{\text{apr}}_{c^{-1}}$,

(b) $H_5^{\mathbb{C}} = \overline{\text{apr}}_{a^{-1}}$.

However, we have no such equality for the approximation operators $H_1^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$. We study the partial order relations with respect to $\leq$ for the pairs 1 – 29 of Table 4.4. The only crucial partial order relations which hold for $H_1^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ are proved in [166]:

Proposition 4.4.4. [166] Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $H_5^{\mathbb{C}} \leq H_1^{\mathbb{C}} \leq H_3^{\mathbb{C}}$,

(b) $H_1^{\mathbb{C}} \leq H_4^{\mathbb{C}} \leq H_2^{\mathbb{C}}$.

We provide counterexamples for the partial order relations which do not hold in Counterexample 5 of Appendix A and in the following two examples.

Example 4.4.5. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{12,13,24,34\}$, then it holds that $\overline{\text{apr}}_1(\{1\}) = \{1\}$ and $\overline{\text{apr}}_2(\{1\}) = \{1,2,3\}$, hence $\overline{\text{apr}}_2 \leq \overline{\text{apr}}_1$ does not hold for $\overline{\text{apr}}_1 \in \{\overline{\text{apr}}_l, \overline{\text{apr}}_{l^{-1}}\}$ and $\overline{\text{apr}}_2 \in \{H_1^{\mathbb{C}}, H_4^{\mathbb{C}}\}$. Moreover, $\overline{\text{apr}}_{S_\cap}(\{1,2,3\}) = \{1,2,3,4\}$ and $H_4^{\mathbb{C}}(\{1,2,3\}) = \{1,2,3\}$, hence, $\overline{\text{apr}}_{S_\cap} \leq H_4^{\mathbb{C}}$ does not hold.

Example 4.4.6. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1,12,23,24,123,234\}$, then we have that $\overline{\text{apr}}_e(\{2\}) = \{2,3,4\}$, $\overline{\text{apr}}_{e^{-1}}(\{2\}) = \{1,2,4\}$ and $H_1^{\mathbb{C}}(\{2\}) = \{1,2,3,4\}$, hence, $H_1^{\mathbb{C}} \leq \overline{\text{apr}}_e$ and $H_1^{\mathbb{C}} \leq \overline{\text{apr}}_{e^{-1}}$ do not hold. Moreover, $\overline{\text{apr}}_{f^{-1}}(\{1\}) = \{1,2,3\}$ and $H_4^{\mathbb{C}}(\{1\}) = \{1\}$, hence, $\overline{\text{apr}}_{f^{-1}} \leq H_4^{\mathbb{C}}$ does not hold.

The Hasse diagram with respect for $\leq$ for the upper approximation operators of pairs 1 – 29 in Table 4.4 can be found in Figure 4.6.

4.5 Framework of Zhao

In [197], Zhao studied seven pairs of dual covering-based approximation operators from a topological point of view. Moreover, Zhao studied the partial order relations between these seven dual pairs. The Hasse diagram of the upper approximation operators can be found in Figure 4.7. In addition, Zhao studied under which conditions some of the pairs coincide with each other. However, in general, they

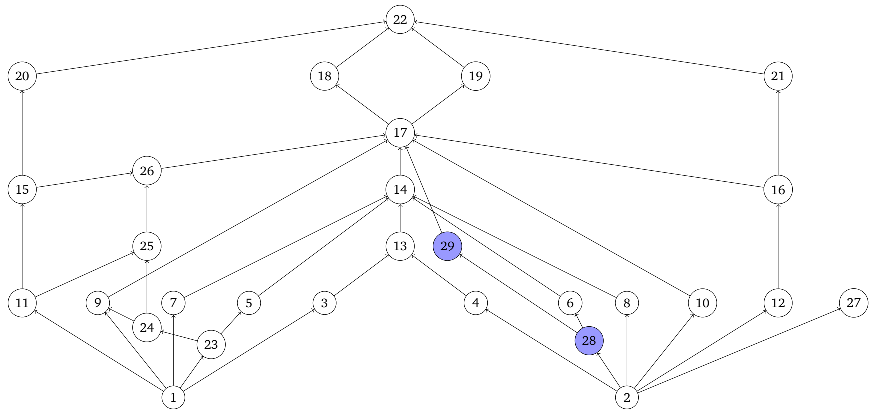


Figure 4.6: Hasse diagram for the upper approximation operators of pairs 1 – 29 in Table 4.4, where we have extended the Hasse diagram presented in Figure 4.5 with pairs 28 and 29.

are all different.

We now want to extend the framework provided in Figure 4.6 with the seven pairs of approximation operators studied by Zhao. By definition, we immediately see the following result:

Proposition 4.5.1. Let $(U, \mathbb{C})$ be a covering approximation space, then it holds that $(l^-, l^+) = (\underline{\text{apr}}_a, \overline{\text{apr}}_a)$.

Hence, the neighborhood operator $N_1^{\mathbb{C}}$ effectively generates the interior and closure operator of the topology induced by $\mathbb{C}$. Moreover, as the following proposition shows, the inverse neighborhood operator $(N_1^{\mathbb{C}})^{-1}$, sometimes called the *complementary neighborhood operator* [106], also has a topological interpretation:

Proposition 4.5.2. Let $(U, \mathbb{C})$ be a covering approximation space and $\mathscr{T}$ the topology induced by $\mathbb{C}$. For $x \in U$ it holds that $\overline{\{x\}} = (N_1^{\mathbb{C}})^{-1}(x)$.

Proof. Assume $x \in U$. From [197] we obtain that the closure of $\{x\}$ with respect

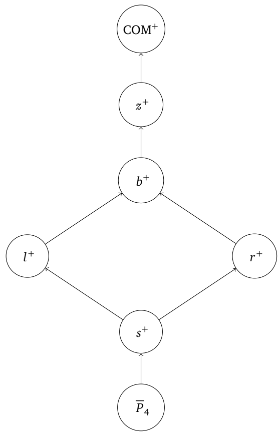


Figure 4.7: Hasse diagram of upper approximation operators studied by Zhao

to the induced topology is given by $U \setminus \left(\bigcup_{K\in\mathbb{C}, x\notin K} K \right)$. Hence, we find

$$\begin{aligned}
\overline{\{x\}} &= U \setminus \left(\bigcup_{K\in\mathbb{C}, x\notin K} K \right) \\
&= \left\{ y \in U \mid y \notin \bigcup_{K\in\mathbb{C}, x\notin K} K \right\} \\
&= \{ y \in U \mid (\forall K \in \mathbb{C})(x \notin K \Rightarrow y \notin K) \} \\
&= \{ y \in U \mid (\forall K \in \mathbb{C})(y \in K \Rightarrow x \in K) \} \\
&= \left\{ y \in U \mid x \in N_1^{\mathbb{C}}(y) \right\} \\
&= (N_1^{\mathbb{C}})^{-1}(x).
\end{aligned}$$

□

From the previous proposition, we obtain the following equality of approximation operators.

Corollary 4.5.3. Let $(U, \mathbb{C})$ be a covering approximation space, then it holds that $(r^-, r^+) = (\underline{\text{apr}}_{a^{-1}}, \overline{\text{apr}}_{a^{-1}})$.

Continuing, it is straightforward to check that the dual pairs $(\underline{P}_4, \overline{P}_4)$, (b^-, b^+), (z^-, z^+) and $(\text{COM}^-, \text{COM}^+)$ are also element-based approximation operators. We will denote the corresponding neighborhood operators by $N^{\mathbb{C}}_{P_4}$, $N^{\mathbb{C}}_b$, $N^{\mathbb{C}}_z$ and $N^{\mathbb{C}}_{\text{COM}}$ respectively, i.e., for x in U,

$$\begin{aligned} N^{\mathbb{C}}_{P_4}(x) &= P^{\mathbb{C}}_x, \\ N^{\mathbb{C}}_b(x) &= N^{\mathbb{C}}_1(x) \cup \overline{\{x\}} \\ N^{\mathbb{C}}_z(x) &= \overline{N^{\mathbb{C}}_1(x)}, \\ N^{\mathbb{C}}_{\text{COM}}(x) &= [x]_{\sim}. \end{aligned}$$

All of the four neighborhood operators are reflexive and symmetric, so they coincide with their inverse neighborhood operators. Moreover, $N^{\mathbb{C}}_{P_4}$ and $N^{\mathbb{C}}_{\text{COM}}$ are equivalence relations and thus are also transitive. Neither $N^{\mathbb{C}}_b$ nor $N^{\mathbb{C}}_z$ are transitive, as the following example demonstrates.

Example 4.5.4. For the covering approximation space $(U, \mathbb{C})$ with $U = \{1,2,3,4\}$ and $\mathbb{C} = \{12, 13, 234\}$, then induced topology $\mathscr{T}$ is given by

$$\mathscr{T} = \{\emptyset, 1234, 12, 13, 234, 1, 2, 3, 123\}.$$

We find, for example, that $N^{\mathbb{C}}_b(2) = N^{\mathbb{C}}_z(2) = \{2,4\}$ and $N^{\mathbb{C}}_b(4) = N^{\mathbb{C}}_z(4) = \{2,3,4\}$. So, $4 \in N^{\mathbb{C}}_b(2) = N^{\mathbb{C}}_z(2)$, but $N^{\mathbb{C}}_b(4) = N^{\mathbb{C}}_z(4) \not\subseteq N^{\mathbb{C}}_b(2) = N^{\mathbb{C}}_z(2)$. Hence, $N^{\mathbb{C}}_b$ and $N^{\mathbb{C}}_z$ are not transitive.

The four neighborhood operators $N^{\mathbb{C}}_b$, $N^{\mathbb{C}}_z$, $N^{\mathbb{C}}_{\text{COM}}$ and $N^{\mathbb{C}}_{P_4}$ do not coincide[3] with one of the neighborhood operators from Table 4.2. Therefore, we can extend Table 4.2 to Table 4.3. We will denote the four new neighborhood operators with the lowercase letters $n-q$. In addition, we can denote the four pairs of approximation operators as follows:

$$\begin{aligned} (\underline{P}_4, \overline{P}_4) &= (\underline{\text{apr}}_n, \overline{\text{apr}}_n), \\ (b^-, b^+) &= (\underline{\text{apr}}_o, \overline{\text{apr}}_o), \\ (z^-, z^+) &= (\underline{\text{apr}}_p, \overline{\text{apr}}_p), \end{aligned}$$

[3]Counterexamples to support this claim can be found later on in the text, when we discuss the partial order relations of the covering-based approximation operators.

$$(\text{COM}^-, \text{COM}^+) \quad = \quad (\underline{\text{apr}}_q, \overline{\text{apr}}_q).$$

Moreover, note that the following example shows that (s^-, s^+) does not meet the criteria of an element-based dual pair of approximation operators.

Example 4.5.5. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{1, 12, 123\}$, then $\mathscr{T}$ is given by

$$\mathscr{T} = \{\emptyset, 1, 2, 123\}.$$

It holds that $N_1^{\mathbb{C}}(1) = \{1\}$, $N_1^{\mathbb{C}}(2) = \{1,2\}$, $N_1^{\mathbb{C}}(3) = \{1,2,3\}$, and $\overline{1} = \{1,2,3\}$, $\overline{2} = \{2,3\}$, $\overline{3} = \{3\}$.

In addition, $s^+(\{1\}) = \{1\}$ and $s^+(\{3\}) = \{3\}$, thus, $s^+(\{1\}) \cup s^+(\{3\}) = \{1,3\}$, yet $s^+(\{1\} \cup \{3\}) = s^+(\{1,3\}) = \{1,2,3\}$. Therefore, by Proposition 4.2.2, s^+ is not an element-based approximation operator.

The pair (s^-, s^+) does not equal any of the covering-based approximation operators of pairs 1 – 29 in Table 4.4. We will now study how to extend the Hasse diagram in Figure 4.6 with the five upper approximations of pairs 30 – 34. From Figure 4.7 it is clear that the following partial order relations hold:

Proposition 4.5.6. [197] Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $\overline{\text{apr}}_n \leq s^+ \leq \overline{\text{apr}}_a$,

(b) $\overline{\text{apr}}_n \leq s^+ \leq \overline{\text{apr}}_{a^{-1}}$.

Hence, the upper approximation operators $\overline{\text{apr}}_n$ and s^+ yield smaller upper approximations than all the upper approximations of pairs 1 – 29 considered in Table 4.4. From Counterexample 5 of Appendix A it is clear that s^+ yields strictly smaller upper approximations than $\overline{\text{apr}}_a$ and $\overline{\text{apr}}_{a^{-1}}$.

Moreover, from Figure 4.7 we also derive following partial order relations:

Proposition 4.5.7. [197] Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $\overline{\text{apr}}_a \leq \overline{\text{apr}}_o \leq \overline{\text{apr}}_p \leq \overline{\text{apr}}_q$,

(b) $\overline{\text{apr}}_{a^{-1}} \leq \overline{\text{apr}}_o \leq \overline{\text{apr}}_p \leq \overline{\text{apr}}_q$.

Table 4.3: Neighborhood operators for $(U, \mathbb{C})$

Group	Operators	Group	Operators
$a.$	$N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$, $N_2^{\mathbb{C}_3}$, $N_1^{\mathbb{C}_\cap}$	$a^{-1}.$	$(N_1^{\mathbb{C}})^{-1}$, $(N_1^{\mathbb{C}_1})^{-1}$, $(N_1^{\mathbb{C}_3})^{-1}$, $(N_2^{\mathbb{C}_3})^{-1}$, $(N_1^{\mathbb{C}_\cap})^{-1}$
$b.$	$N_3^{\mathbb{C}_3}$	$b^{-1}.$	$(N_3^{\mathbb{C}_3})^{-1}$
$c.$	$N_2^{\mathbb{C}}$, $N_2^{\mathbb{C}_1}$	$c^{-1}.$	$(N_2^{\mathbb{C}})^{-1}$, $(N_2^{\mathbb{C}_1})^{-1}$
$d.$	$N_3^{\mathbb{C}_1}$	$d^{-1}.$	$(N_3^{\mathbb{C}_1})^{-1}$
$e.$	$N_2^{\mathbb{C}_\cap}$	$e^{-1}.$	$(N_2^{\mathbb{C}_\cap})^{-1}$
$f.$	$N_3^{\mathbb{C}}$, $N_1^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_\cap}$	$f^{-1}.$	$(N_3^{\mathbb{C}})^{-1}$, $(N_1^{\mathbb{C}_2})^{-1}$, $(N_3^{\mathbb{C}_2})^{-1}$, $(N_3^{\mathbb{C}_\cap})^{-1}$
$g.$	$N_4^{\mathbb{C}_3}$, $(N_4^{\mathbb{C}_3})^{-1}$		
$h.$	$N_4^{\mathbb{C}_1}$, $(N_4^{\mathbb{C}_1})^{-1}$		
$i.$	$N_1^{\mathbb{C}_4}$	$i^{-1}.$	$(N_1^{\mathbb{C}_4})^{-1}$
$j.$	$N_4^{\mathbb{C}}$, $N_2^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_\cap}$, $(N_4^{\mathbb{C}})^{-1}$, $(N_2^{\mathbb{C}_2})^{-1}$, $(N_4^{\mathbb{C}_2})^{-1}$, $(N_4^{\mathbb{C}_\cap})^{-1}$		
$k.$	$N_2^{\mathbb{C}_4}$	$k^{-1}.$	$(N_2^{\mathbb{C}_4})^{-1}$
$l.$	$N_3^{\mathbb{C}_4}$	$l^{-1}.$	$(N_3^{\mathbb{C}_4})^{-1}$
$m.$	$N_4^{\mathbb{C}_4}$, $(N_4^{\mathbb{C}_4})^{-1}$		
$n.$	$N_{P_4}^{\mathbb{C}}$, $(N_{P_4}^{\mathbb{C}})^{-1}$		
$o.$	$N_b^{\mathbb{C}}$, $(N_b^{\mathbb{C}})^{-1}$		
$p.$	$N_z^{\mathbb{C}}$, $(N_z^{\mathbb{C}})^{-1}$		
$q.$	$N_{\mathrm{COM}}^{\mathbb{C}}$, $(N_{\mathrm{COM}}^{\mathbb{C}})^{-1}$		

Again, from Counterexample 5 of Appendix A it is clear that $\overline{\text{apr}}_o$ yields strictly larger upper approximations than $\overline{\text{apr}}_a$ and $\overline{\text{apr}}_{a^{-1}}$.

In addition, the following partial order relations hold:

Proposition 4.5.8. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $\overline{\text{apr}}_o \leq \overline{\text{apr}}_g$,

(b) $\overline{\text{apr}}_g \leq \overline{\text{apr}}_q$,

(c) $\overline{\text{apr}}_p \leq \overline{\text{apr}}_m$.

Proof. Let $(U, \mathbb{C})$ be a covering approximation space, $\mathscr{T}$ the topology induced by $\mathbb{C}$ and $X \subseteq U$.

(a) Let $x \in U$ with $x \in \overline{\text{apr}}_o(X)$, then $N_b^{\mathbb{C}}(x) \cap X \neq \emptyset$. We shall prove that $N_b^{\mathbb{C}}(x) \subseteq N_4^{\mathbb{C}_3}(x)$. Let $y \in U$ with $y \in N_b^{\mathbb{C}}(x)$, then either $y \in N_1^{\mathbb{C}}(x)$ or $y \in (N_1^{\mathbb{C}})^{-1}(x)$. In both cases it holds that $y \in N_4^{\mathbb{C}_3}(x)$. Hence, $N_4^{\mathbb{C}_3}(x) \cap X \neq \emptyset$, and thus, $x \in \overline{\text{apr}}_g(X)$.

(b) Let $x \in U$ with $x \in \overline{\text{apr}}_g(X)$, then $N_4^{\mathbb{C}_3}(x) \cap X \neq \emptyset$. We shall prove that $N_4^{\mathbb{C}_3}(x) \subseteq N_{\text{COM}}^{\mathbb{C}}(x)$. Let $y \in U$ with $y \in N_4^{\mathbb{C}_3}(x)$, then there exists $K \in \mathbb{C}_3$ such that $x, y \in K$. In other words, there exists $z \in U$ such that $x, y \in N_1^{\mathbb{C}}(z)$. Consider the topology $\mathscr{T}$ induced by $\mathbb{C}$ and $V = \{x, y, z\}$. It is clear that the topological subspace $(V, \mathscr{T}_V)$ of $(U, \mathscr{T})$ is connected, since $N_1^{\mathbb{C}}(z) \cap V = \{x, y, z\}$ is the smallest set in T_V which contains the element z. As a consequence, $[x]_\sim = [y]_\sim$ and hence $y \in N_{\text{COM}}^{\mathbb{C}}(x)$. We conclude $N_{\text{COM}}^{\mathbb{C}}(x) \cap X \neq \emptyset$ and thus, $x \in \overline{\text{apr}}_q(X)$.

(c) Let $x \in U$ with $x \in \overline{\text{apr}}_p(X)$, then $N_z^{\mathbb{C}}(x) \cap X \neq \emptyset$. We shall prove that $(N_4^{\mathbb{C}_4}(x))^c \subseteq (N_z^{\mathbb{C}}(x))^c$. Assume $y \in (N_4^{\mathbb{C}_4}(x))^c$. We want to prove that $y \in (N_z^{\mathbb{C}}(x))^c$, i.e.,

$$\begin{aligned} y \in (N_z^{\mathbb{C}}(x))^c \quad &\Leftrightarrow \quad y \notin \bigcap\{Y \in \mathscr{T}^c \mid N_1^{\mathbb{C}}(x) \subseteq Y\} \\ &\Leftrightarrow \quad y \notin \bigcap\{X^c \mid X \in \mathscr{T}, N_1^{\mathbb{C}}(x) \subseteq X^c\} \\ &\Leftrightarrow \quad y \notin \bigcap\{X^c \mid X \in \mathscr{T}, N_1^{\mathbb{C}}(x) \cap X = \emptyset\} \\ &\Leftrightarrow \quad y \in \bigcup\{X \in \mathscr{T} \mid N_1^{\mathbb{C}}(x) \cap X = \emptyset\}. \end{aligned}$$

By definition, it holds that $N_1^{\mathbb{C}}(y) \in \mathscr{T}$ and $y \in N_1^{\mathbb{C}}(y)$. Since $y \in (N_4^{\mathbb{C}_4}(x))^c$, it holds that $N_4^{\mathbb{C}}(x) \cap N_4^{\mathbb{C}}(y) = \emptyset$, hence, $N_1^{\mathbb{C}}(x) \cap N_1^{\mathbb{C}}(y) = \emptyset$. Therefore, $y \in \bigcup\{X \in \mathscr{T} \mid N_1^{\mathbb{C}}(x) \cap X = \emptyset\}$ and thus, $y \in (N_z^{\mathbb{C}}(x))^c$. We conclude that $N_z^{\mathbb{C}}(x) \subseteq N_4^{\mathbb{C}_4}(x)$, hence, $N_4^{\mathbb{C}_4}(x) \cap X \neq \emptyset$ and thus, $x \in \overline{\text{apr}}_m(X)$.

□

No other partial order relations hold. Counterexamples can be found in Counterexamples 5 and 6 of Appendix A and the following two examples:

Example 4.5.9. Let $U = \{1,2,3,4\}$ with $\mathbb{C} = \{12, 13, 234\}$, then $\mathscr{T}$ is given by

$$\mathscr{T} = \{\emptyset, 1234, 12, 13, 234, 1, 2, 3, 123\}.$$

For $\overline{\text{apr}}_1 \in \{\overline{\text{apr}}_b, \overline{\text{apr}}_{b^{-1}}\}$ and $\overline{\text{apr}}_2 \in \{\overline{\text{apr}}_o, \overline{\text{apr}}_p\}$ it holds that $\overline{\text{apr}}_1(\{2\}) = \{2,3,4\}$ and $\overline{\text{apr}}_2(\{2\}) = \{2,4\}$. Hence, $\overline{\text{apr}}_1 \leq \overline{\text{apr}}_2$ does not hold.

In addition, $\overline{\text{apr}} \in \{\overline{\text{apr}}_o, \overline{\text{apr}}_p, \overline{\text{apr}}_q\}$ it holds that $\overline{\text{apr}}(\{2,3,4\}) = \{2,3,4\}$ and $\overline{\text{apr}}'_{\mathbb{C}}(\{2,3,4\}) = \{1,2,3,4\}$. Hence, $\overline{\text{apr}}'_{\mathbb{C}} \leq \overline{\text{apr}}$ does not hold.

Example 4.5.10. Let $U = \{1,2,3,4\}$ with $\mathbb{C} = \{1, 3, 4, 234\}$, then $\mathscr{T}$ is given by

$$\mathscr{T} = \{\emptyset, 1234, 1, 3, 4, 234, 13, 14, 34, 134\}.$$

It holds that $\overline{\text{apr}}_c(\{2\}) = \overline{\text{apr}}_e(\{2\}) = \{2\}$ and $\overline{\text{apr}}_o(\{2\}) = \{2,3,4\}$ and moreover, $\overline{\text{apr}}_{c^{-1}}(\{3\}) = \overline{\text{apr}}_{e^{-1}}(\{3\}) = \{3\}$ and $\overline{\text{apr}}_o(\{3\}) = \{2,3\}$. Hence, $\overline{\text{apr}}_o \leq \overline{\text{apr}}$ does not hold for $\overline{\text{apr}} \in \{\overline{\text{apr}}_c, \overline{\text{apr}}_{c^{-1}}, \overline{\text{apr}}_e, \overline{\text{apr}}_{e^{-1}}\}$.

Hence, the Hasse diagram with respect to $\leq$ for the upper approximation operators of pairs 1 – 34 of Table 4.4 is given in Figure 4.8.

Note that by Proposition 4.1.1 we easily obtain the Hasse diagram with respect to $\preceq$ for the neighborhood operators in Table 4.3 from Figure 4.8 in Figure 4.9.

4.6 Framework of Samanta and Chakraborty

In [142, 143], Samanta and Chakraborty studied different covering-based approximation operators. By definition, many of these operators coincide with pairs 1 – 34 of Table 4.4:

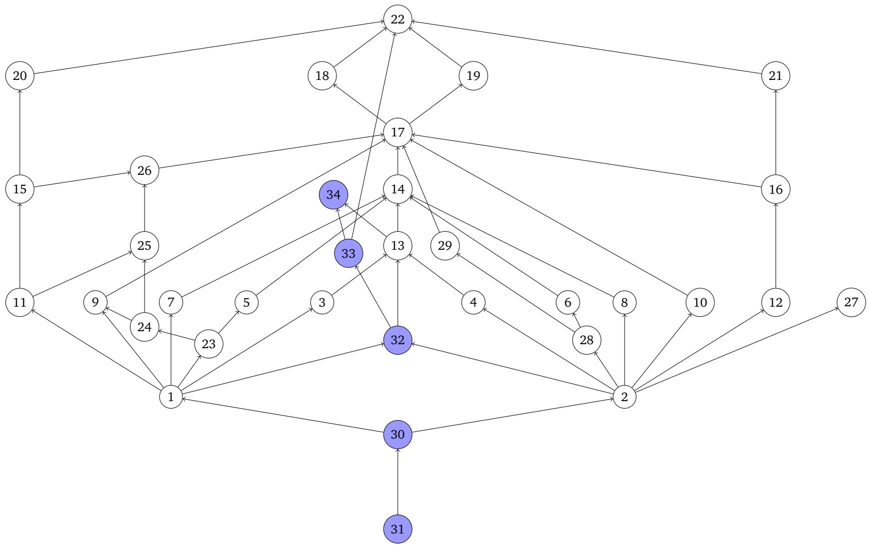


Figure 4.8: Hasse diagram for the upper approximation operators of pairs 1 – 34 in Table 4.4, where we have extended the Hasse diagram presented in Figure 4.6 with pairs 30 – 34.

- The pair $(\underline{P}_1, \overline{P}_1)$ coincides with pair 17.
- The pair $(\underline{P}_2, \overline{P}_2)$ coincides with pair 26.
- The pairs $(\underline{P}_3, \overline{P}_3)$ and $(\underline{C}_1, \overline{C}_1)$ coincide with pair 23.
- The pair $(\underline{P}_4, \overline{P}_4)$ coincides with pair 31.
- The pair $(\underline{C}_2, \overline{C}_2)$ coincides with pair 1.
- The pair $(\underline{C}_4, \overline{C}_4)$ coincides with pair 13.
- The pair $(\underline{C}_5, \overline{C}_5)$ coincides with pair 2.
- The pair $(\underline{C}^*, \overline{C}^*)$ coincides with pair 28.
- The pair $(\underline{C}^-, \overline{C}^-)$ coincides with pair 17.

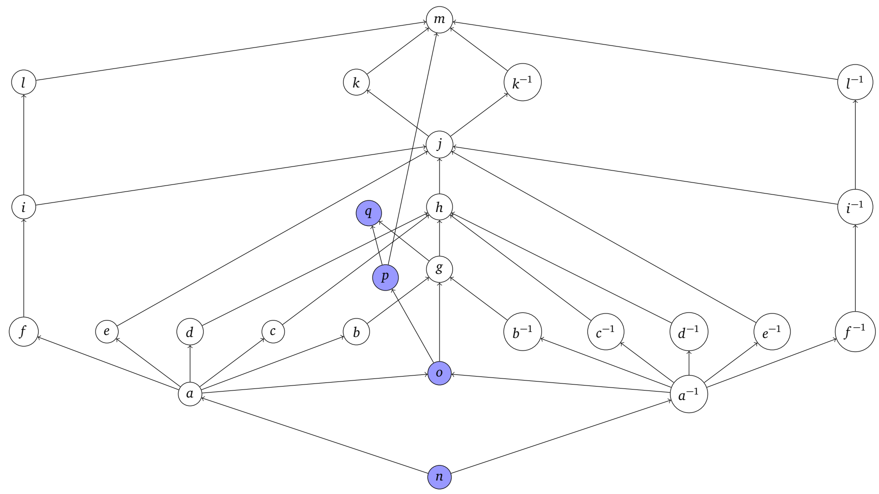


Figure 4.9: Hasse diagram for the neighborhood operators in Table 4.3

- The pair $(\underline{C}^{\#}, \overline{C}^{\#})$ coincides with pair 6.
- The pair $(\underline{C}^{@}, \overline{C}^{@})$ coincides with pair 29.

Moreover, we have the following equalities between approximation operators studied by Samanta and Chakraborty and pairs 1 – 34 of Table 4.4.

Proposition 4.6.1. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $(\underline{C}^{+}, \overline{C}^{+}) = (L_5^{\mathbb{C}}, H_5^{\mathbb{C}})$,

(b) $(\underline{C}^{Gr}, \overline{C}^{Gr}) = (\underline{\text{apr}}'_{\mathbb{C}}, \overline{\text{apr}}'_{\mathbb{C}})$.

Proof. Let $X \subseteq U$.

(a)

$$\begin{aligned}
& \underline{\text{apr}}'_{\mathbb{C}}(X) \\
= \; & \{x \in U \mid \exists K \in \mathbb{C} \colon x \in K \wedge K \subseteq X\} \\
= \; & \{x \in U \mid \exists K \in \mathbb{C} \colon N_1^{\mathbb{C}}(x) \subseteq K \wedge K \subseteq X\} \\
= \; & \{x \in U \mid \exists K \in \mathbb{C} \colon (\exists y \in U \colon x \in N_1^{\mathbb{C}}(y) \wedge N_1^{\mathbb{C}}(y) \subseteq K) \wedge K \subseteq X\}
\end{aligned}$$

$$
\begin{aligned}
&= \{x \in U \mid \exists y \in U : x \in N_1^{\mathbb{C}}(y) \wedge (\exists K \in \mathbb{C} : N_1^{\mathbb{C}}(y) \subseteq K \wedge K \subseteq X)\} \\
&= \{x \in U \mid \exists y \in U : x \in N_1^{\mathbb{C}}(y) \wedge (\exists K \in \mathbb{C} : y \in K \wedge K \subseteq X)\} \\
&= \{x \in U \mid \exists y \in \underline{C}_1(X) : x \in N_1^{\mathbb{C}}(y)\} \\
&= \bigcup \{N_1^{\mathbb{C}}(y) \mid y \in \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\overline{C}^{+}(X) &= \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \cup \left(\bigcup \{N_1^{\mathbb{C}}(y) \mid y \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\}\right) \\
&= \bigcup \{N_1^{\mathbb{C}}(y) \mid y \in X\} \\
&= H_5^{\mathbb{C}}(X).
\end{aligned}
$$

(b) By definition, we have that $\underline{C}^{Gr} = \underline{\mathrm{apr}}'_{\mathbb{C}}$. Moreover, the following holds for $X \subseteq U$:

$$
\begin{aligned}
\overline{C}^{Gr}(X) &= \left(\overline{\mathrm{apr}}''_{\mathbb{C}}(X)\right) \setminus \left(\underline{C}_{Gr}(X^{\mathrm{c}})\right) \\
&= \overline{\mathrm{apr}}''_{\mathbb{C}}(X) \cap (\underline{C}_{Gr}(X^{\mathrm{c}}))^{\mathrm{c}} \\
&= \overline{\mathrm{apr}}''_{\mathbb{C}}(X) \cap (\underline{\mathrm{apr}}'_{\mathbb{C}}(X^{\mathrm{c}}))^{\mathrm{c}} \\
&= \overline{\mathrm{apr}}''_{\mathbb{C}}(X) \cap \overline{\mathrm{apr}}'_{\mathbb{C}}(X) \\
&= \overline{\mathrm{apr}}'_{\mathbb{C}}(X),
\end{aligned}
$$

since $\overline{\mathrm{apr}}'_{\mathbb{C}} \leq \overline{\mathrm{apr}}''_{\mathbb{C}}$.

□

We conclude that the pair $(\underline{C}^{+}, \overline{C}^{+})$ coincides with pair 2 and the pair $(\underline{C}^{Gr}, \overline{C}^{Gr})$ coincides with pair 23. The only approximation operators discussed by Samanta and Chakraborty which are not considered in the pairs 1 – 34 of Table 4.4 are the pairs $(\underline{C}_3, \overline{C}_3)$ and $(\underline{C}^{\%}, \overline{C}^{\%})$. In the following, we will discuss partial order relations for these two pairs of approximation operators. In addition, we will also illustrate that both pairs lack an important property of approximation operators, namely the inclusion property.

First, we discuss the pair of dual approximation operators $(\underline{C}_3, \overline{C}_3)$. For this pair, the only essential partial order relation which holds is the following one:

Proposition 4.6.2. Let $(U, \mathbb{C})$ a covering approximation space, then $\overline{C}_3 \leq \overline{\text{apr}}_a$.

Proof. Let $X \subseteq U$ and $x \in \overline{C}_3(X)$, then for all $y \in U$ it holds that if $y \in N_1^{\mathbb{C}}(x)$, then $N_1^{\mathbb{C}}(y) \cap X \neq \emptyset$. As $x \in N_1^{\mathbb{C}}(x)$, we have that $N_1^{\mathbb{C}}(x) \cap X \neq \emptyset$, hence, $x \in \overline{\text{apr}}_a(X)$. □

By the transitivity of $\leq$, $\overline{C}_3$ provides smaller upper approximations than all upper approximation operators which are larger than $\overline{\text{apr}}_a$. From Counterexample 5 of Appendix A we determine by $X = \{2\}$ that $\overline{\text{apr}} \leq \overline{C}_3$ does not hold for any upper approximation operator of pairs 1 – 34 of Table 4.4 or for $\overline{C}^{\%}$ since $\overline{C}_3(\{2\}) = \emptyset \not\supseteq \{2\}$. Moreover, we conclude from this example that the pair $(\underline{C}_3, \overline{C}_3)$ does not satisfy the inclusion property given by $\forall X \subseteq U: \underline{\text{apr}}(X) \subseteq X \subseteq \overline{\text{apr}}(X)$ [143]. In addition, from Counterexample 5 of Appendix A and the following two examples, we determine that $\overline{C}_3 \leq \overline{\text{apr}}$ does not hold for the upper approximation operator of pairs 2, 4, 6, 8, 10, 12, 16, 21, 27 – 31 and $\overline{C}^{\%}$.

Example 4.6.3. Let $U = \{1,2,3,4,5,6\}$ and $\mathbb{C} = \{123, 145, 26\}$, then we obtain that $\overline{\text{apr}}_{b^{-1}}(\{1\}) = \{1\}$, $\overline{\text{apr}}_{l^{-1}}(\{1\}) = \{1,2,3\}$ and $\overline{C}_3(\{1\}) = \{1,4,5\}$, hence, $\overline{C}_3 \leq \overline{\text{apr}}_{b^{-1}}$ and $\overline{C}_3 \leq \overline{\text{apr}}_{l^{-1}}$ do not hold.

Example 4.6.4. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1, 2, 23, 14\}$, then $\overline{C}_3(\{2\}) = \{2,3\}$ and $\overline{\text{apr}}_{c^{-1}}(\{2\}) = \overline{\text{apr}}_{e^{-1}}(\{2\}) = \{2\}$, hence, $\overline{C}_3 \leq \overline{\text{apr}}_{c^{-1}}$ and $\overline{C}_3 \leq \overline{\text{apr}}_{e^{-1}}$ do not hold.

Next, we discuss the pair $(\underline{C}^{\%}, \overline{C}^{\%})$. For convenience, we introduce the following notation: for $X \subseteq U$,

$$H^{\%}(X) = \left(\bigcup \left\{ \bigcup \{N_4^{\mathbb{C}}(y) \mid y \in U \setminus N_4^{\mathbb{C}}(x)\} \mid x \in X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) \right\}\right)^{c}, \tag{4.1}$$

hence, $\overline{C}^{\%}(X) = \underline{\text{apr}}'_{\mathbb{C}}(X) \cup H^{\%}(X)$. Before we start the discussion on the partial order relations of the pair $(\underline{C}^{\%}, \overline{C}^{\%})$, we make the following remark.

Remark 4.6.5. Let $(U, \mathbb{C})$ be a covering approximation space and $X \subseteq U$ such that $\underline{\text{apr}}'_{\mathbb{C}}(X) = X$. As $X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) = \emptyset$, it holds that $H^{\%}(X) = U$, hence, we obtain that $\overline{C}^{\%}(X) = X \cup U = U$. As we assume that the authors of [99] have drawn inspiration from the operators $H_1^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ when introducing this upper approximation operator, it should hold in this case that $\overline{C}^{\%}(X) = X$. Therefore, we rewrite the definition of $\overline{C}^{\%}$ as follows: let $X \subseteq U$, then

$$\overline{C}^{\%}(X) = \begin{cases} X & X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X) = \emptyset, \\ \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \cup H^{\%}(X) & X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \neq \emptyset. \end{cases} \tag{4.2}$$

Interpreting the operator $\overline{C}^{\%}$ as in Eq. (4.2), we prove the following characterization for the auxiliary operator $H^{\%}$:

Proposition 4.6.6. Let $(U, \mathbb{C})$ be a covering approximation space and $X \subseteq U$ such that $X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \neq \emptyset$, then

$$H^{\%}(X) = \bigcap \{(N_1^{\mathbb{C}_4})^{-1}(x) \mid x \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\}. \tag{4.3}$$

Proof. Let $X \subseteq U$ such that $X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X) \neq \emptyset$ and $x \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)$, then

$$\begin{aligned} & z \in \bigcup \{N_4^{\mathbb{C}}(y) \mid y \in U \setminus N_4^{\mathbb{C}}(x)\} \\ \Leftrightarrow\ & \exists y \in U : y \notin N_4^{\mathbb{C}}(x) \wedge z \in N_4^{\mathbb{C}}(y) \\ \Leftrightarrow\ & \exists y \in U : x \notin N_4^{\mathbb{C}}(y) \wedge z \in N_4^{\mathbb{C}}(y) \\ \Leftrightarrow\ & \exists K \in \mathbb{C}_4 : x \notin K \wedge z \in K \\ \Leftrightarrow\ & x \notin N_1^{\mathbb{C}_4}(z) \\ \Leftrightarrow\ & z \notin (N_1^{\mathbb{C}_4})^{-1}(x). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} H^{\%}(X) &= \left(\bigcup \{((N_1^{\mathbb{C}_4})^{-1}(x))^{c} \mid x \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\}\right)^{c} \\ &= \bigcap \{(N_1^{\mathbb{C}_4})^{-1}(x) \mid x \in X \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(X)\}. \end{aligned}$$

□

Given this characterization for $H^{\%}$, we have the following two essential partial order relations for $\overline{C}^{\%}$:

Proposition 4.6.7. Let $(U, \mathbb{C})$ be a covering approximation space, then

(a) $\overline{C}^{\%} \leq \overline{\text{apr}}_i$,

(b) $\overline{C}^{\%} \leq H_4^{\mathbb{C}}$.

Proof. (a) Let $X \subseteq U$. If $X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) = \emptyset$, then $\overline{C}^{\%}(X) = X \subseteq \overline{\text{apr}}_i(X)$. On the other hand, if $X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) \neq \emptyset$, then we need to prove that

$$\overline{C}^{\%}(X) = \underline{\text{apr}}'_{\mathbb{C}}(X) \cup H^{\%}(X) \subseteq \overline{\text{apr}}_i(X),$$

i.e., we need to prove that $H^{\%}(X) \subseteq \overline{\text{apr}}_i(X)$. Assume $z \in H^{\%}(X)$, then for all $x \in X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)$ it holds that $z \in (N_1^{\mathbb{C}_4})^{-1}(x)$, hence, $x \in N_1^{\mathbb{C}_4}(z)$. Therefore, $N_1^{\mathbb{C}_4}(z) \cap X \neq \emptyset$, and thus, $H^{\%}(X) \subseteq \overline{\text{apr}}_i(X)$.

(b) This claim was already proven in [99]. However, we provide an alternative proof here, as the proof in [99] is rather concise.

Let $X \subseteq U$. If $X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) = \emptyset$, then $\overline{C}^{\%}(X) = X \subseteq H_4^{\mathbb{C}}(X)$. On the other hand, if $X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X) \neq \emptyset$, then we need to prove that

$$\overline{C}^{\%}(X) = \underline{\text{apr}}'_{\mathbb{C}}(X) \cup H^{\%}(X) \subseteq H_4^{\mathbb{C}}(X),$$

i.e., we need to prove that

$$H^{\%}(X) \subseteq \bigcup \{K \in \mathbb{C} \mid K \cap (X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)) \neq \emptyset\}.$$

Let $z \in H^{\%}(X)$, then for all $x \in X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)$ it holds that $z \in (N_1^{\mathbb{C}_4})^{-1}(x)$, hence, $x \in N_1^{\mathbb{C}_4}(z)$. As $N_1^{\mathbb{C}_4} \preceq N_4^{\mathbb{C}}$, it holds that $x \in N_4^{\mathbb{C}}(z)$, hence, there exists a $K \in \mathbb{C}$ with $x, z \in K$. We conclude that $z \in \bigcup \{K \in \mathbb{C} \mid K \cap (X \setminus \underline{\text{apr}}'_{\mathbb{C}}(X)) \neq \emptyset\}$. □

All the other partial order relations which hold for $\overline{C}^{\%}$ follow by the transitivity of $\leq$. Counterexamples can be found in Counterexample 5 of Appendix A and in the following three examples:

Example 4.6.8. Let $U = \{1,2,3,4\}$ and $\mathbb{C} = \{1,12,23,24,123,234\}$, then we obtain that $\overline{\text{apr}}_{e^{-1}}(\{2\}) = \{1,2,4\}$ and $\overline{C}^{\%}(\{2\}) = \{1,2,3,4\})$. Moreover,

(a) $\overline{\text{apr}}_d(\{3\}) = \overline{\text{apr}}_e(\{3\}) = \{3\}$,

(b) $\overline{\text{apr}}_{c^{-1}}(\{3\}) = \overline{\text{apr}}_h(\{3\}) = H_1^{\mathbb{C}}(\{3\}) = \{2,3\}$,

(c) $\overline{C}^{\%}(\{3\}) = \{1,2,3,4\})$.

Hence, $\overline{C}^{\%} \leq \overline{\text{apr}}$ does not hold for

$$\overline{\text{apr}} \in \{\overline{\text{apr}}_{c^{-1}}, \overline{\text{apr}}_d, \overline{\text{apr}}_e, \overline{\text{apr}}_{e^{-1}}, \overline{\text{apr}}_h, H_1^{\mathbb{C}}\}.$$

Example 4.6.9. Let $U = \{1,2,3,4,5,6\}$ and $\mathbb{C} = \{123, 145, 26\}$, then we obtain that $\overline{\text{apr}}_{l^{-1}}(\{1\}) = \{1,2,3\}$ and $\overline{C}^{\%}(\{1\}) = \{1,3,4,5\}$, hence, $\overline{C}^{\%} \leq \overline{\text{apr}}_{l^{-1}}$ does not hold.

Example 4.6.10. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{12, 23\}$, then we obtain that

$$\overline{P}_4(\{1,3\}) = s^+(\{1,3\}) = \{1,3\}$$

and $\overline{C}^{\%}(\{1,3\}) = \emptyset$. Hence, $\overline{P}_4 \leq \overline{C}^{\%}$ and $s^+ \leq \overline{C}^{\%}$ do not hold.

The previous example also illustrates that the pair $(\underline{C}^{\%}, \overline{C}^{\%})$ does not satisfy the inclusion property.

We now conclude this section. In Table 4.4, we provide an overview of all pairs of dual covering-based approximation operators considered in this chapter. The Hasse diagram with respect to $\leq$ of the upper approximation operators listed in Table 4.4 is presented in Figure 4.10. Note that we have added the identity function $\text{id}\colon \mathscr{P}(U) \to \mathscr{P}(U)\colon X \mapsto X$ to Figure 4.10, to illustrate that the two pairs $(\underline{C}_3, \overline{C}_3)$ and $(\underline{C}^{\%}, \overline{C}^{\%})$ do not satisfy the inclusion property (INC). This property is satisfied by all the other pairs of dual approximation operators, as we have the following observation: let $(U, \mathbb{C})$ be a covering approximation space, $X \subseteq U$ and $x \in X$, then since $P_x^{\mathbb{C}} \cap X \neq \emptyset$, it holds that $x \in \overline{P}_4(X)$, i.e., $\text{id} \leq \overline{P}_4$.

4.7 Properties of covering-based approximation operators

In this section, we evaluate for each pair of covering-based approximation operators in Table 4.4 which properties the pair satisfies. From Table 2.1, we discuss the

Table 4.4: Overview of covering-based rough set approximation operators

No.	Pairs	No.	Pairs
1	$(\underline{\mathrm{apr}}_a,\overline{\mathrm{apr}}_a)$, $(\underline{\mathrm{apr}}'_{\mathbb{C}_3},\overline{\mathrm{apr}}'_{\mathbb{C}_3})$ $(L_6^{\mathbb{C}},H_6^{\mathbb{C}})$, (l^-,l^+), $(\underline{C}_2,\overline{C}_2)$	23	$(\underline{\mathrm{apr}}'_{\mathbb{C}},\overline{\mathrm{apr}}'_{\mathbb{C}})$, $(\underline{\mathrm{apr}}'_{\mathbb{C}_1},\overline{\mathrm{apr}}'_{\mathbb{C}_1})$ $(\underline{\mathrm{apr}}_{S_\cup},\overline{\mathrm{apr}}_{S_\cup})$, $(\underline{P}_3,\overline{P}_3)$ $(\underline{C}_1,\overline{C}_1)$, $(\underline{C}^{Gr},\overline{C}^{Gr})$
2	$(\underline{\mathrm{apr}}_{a^{-1}},\overline{\mathrm{apr}}_{a^{-1}})$, $(L_5^{\mathbb{C}},H_5^{\mathbb{C}})$ (r^-,r^+), $(\underline{C}_5,\overline{C}_5)$, $(\underline{C}^+,\overline{C}^+)$	24	$(\underline{\mathrm{apr}}'_{\mathbb{C}_\cap},\overline{\mathrm{apr}}'_{\mathbb{C}_\cap})$
3	$(\underline{\mathrm{apr}}_b,\overline{\mathrm{apr}}_b)$	25	$(\underline{\mathrm{apr}}'_{\mathbb{C}_2},\overline{\mathrm{apr}}'_{\mathbb{C}_2})$
4	$(\underline{\mathrm{apr}}_{b^{-1}},\overline{\mathrm{apr}}_{b^{-1}})$	26	$(\underline{\mathrm{apr}}'_{\mathbb{C}_4},\overline{\mathrm{apr}}'_{\mathbb{C}_4})$, $(\underline{P}_2,\overline{P}_2)$
5	$(\underline{\mathrm{apr}}_c,\overline{\mathrm{apr}}_c)$	27	$(\underline{\mathrm{apr}}_{S_\cap},\overline{\mathrm{apr}}_{S_\cap})$
6	$(\underline{\mathrm{apr}}_{c^{-1}},\overline{\mathrm{apr}}_{c^{-1}})$, $(L_3^{\mathbb{C}},H_3^{\mathbb{C}})$, $(\underline{C}^\#,\overline{C}^\#)$	28	$(L_1^{\mathbb{C}},H_1^{\mathbb{C}})$,$(\underline{C}^*,\overline{C}^*)$
7	$(\underline{\mathrm{apr}}_d,\overline{\mathrm{apr}}_d)$	29	$(L_4^{\mathbb{C}},H_4^{\mathbb{C}})$, $(\underline{C}^@,\overline{C}^@)$
8	$(\underline{\mathrm{apr}}_{d^{-1}},\overline{\mathrm{apr}}_{d^{-1}})$	30	(s^-,s^+)
9	$(\underline{\mathrm{apr}}_e,\overline{\mathrm{apr}}_e)$	31	$(\underline{\mathrm{apr}}_n,\overline{\mathrm{apr}}_n)$, $(\underline{P}_4,\overline{P}_4)$
10	$(\underline{\mathrm{apr}}_{e^{-1}},\overline{\mathrm{apr}}_{e^{-1}})$	32	$(\underline{\mathrm{apr}}_o,\overline{\mathrm{apr}}_o)$, (b^-,b^+)
11	$(\underline{\mathrm{apr}}_f,\overline{\mathrm{apr}}_f)$	33	$(\underline{\mathrm{apr}}_p,\overline{\mathrm{apr}}_p)$, (z^-,z^+)
12	$(\underline{\mathrm{apr}}_{f^{-1}},\overline{\mathrm{apr}}_{f^{-1}})$	34	$(\underline{\mathrm{apr}}_q,\overline{\mathrm{apr}}_q)$, $(\mathrm{COM}^-,\mathrm{COM}^+)$
13	$(\underline{\mathrm{apr}}_g,\overline{\mathrm{apr}}_g)$, $(\underline{\mathrm{apr}}''_{\mathbb{C}_3},\overline{\mathrm{apr}}''_{\mathbb{C}_3})$ $(L_7^{\mathbb{C}},H_7^{\mathbb{C}})$, $(\underline{C}_4,\overline{C}_4)$	35	$(\underline{C}_3,\overline{C}_3)$
14	$(\underline{\mathrm{apr}}_h,\overline{\mathrm{apr}}_h)$, $(\underline{\mathrm{apr}}''_{\mathbb{C}_1},\overline{\mathrm{apr}}''_{\mathbb{C}_1})$	36	$(\underline{C}^\%,\overline{C}^\%)$
15	$(\underline{\mathrm{apr}}_i,\overline{\mathrm{apr}}_i)$		
16	$(\underline{\mathrm{apr}}_{i^{-1}},\overline{\mathrm{apr}}_{i^{-1}})$		
17	$(\underline{\mathrm{apr}}_j,\overline{\mathrm{apr}}_j)$, $(\underline{\mathrm{apr}}''_{\mathbb{C}},\overline{\mathrm{apr}}''_{\mathbb{C}})$, $(\underline{\mathrm{apr}}''_{\mathbb{C}_2},\overline{\mathrm{apr}}''_{\mathbb{C}_2})$, $(\underline{\mathrm{apr}}''_{\mathbb{C}_\cap},\overline{\mathrm{apr}}''_{\mathbb{C}_\cap})$ $(L_2^{\mathbb{C}},H_2^{\mathbb{C}})$, $(\underline{P}_1,\overline{P}_1)$, $(\underline{C}^-,\overline{C}^-)$		
18	$(\underline{\mathrm{apr}}_k,\overline{\mathrm{apr}}_k)$		
19	$(\underline{\mathrm{apr}}_{k^{-1}},\overline{\mathrm{apr}}_{k^{-1}})$		
20	$(\underline{\mathrm{apr}}_l,\overline{\mathrm{apr}}_l)$		
21	$(\underline{\mathrm{apr}}_{l^{-1}},\overline{\mathrm{apr}}_{l^{-1}})$		
22	$(\underline{\mathrm{apr}}_m,\overline{\mathrm{apr}}_m)$, $(\underline{\mathrm{apr}}''_{\mathbb{C}_4},\overline{\mathrm{apr}}''_{\mathbb{C}_4})$		

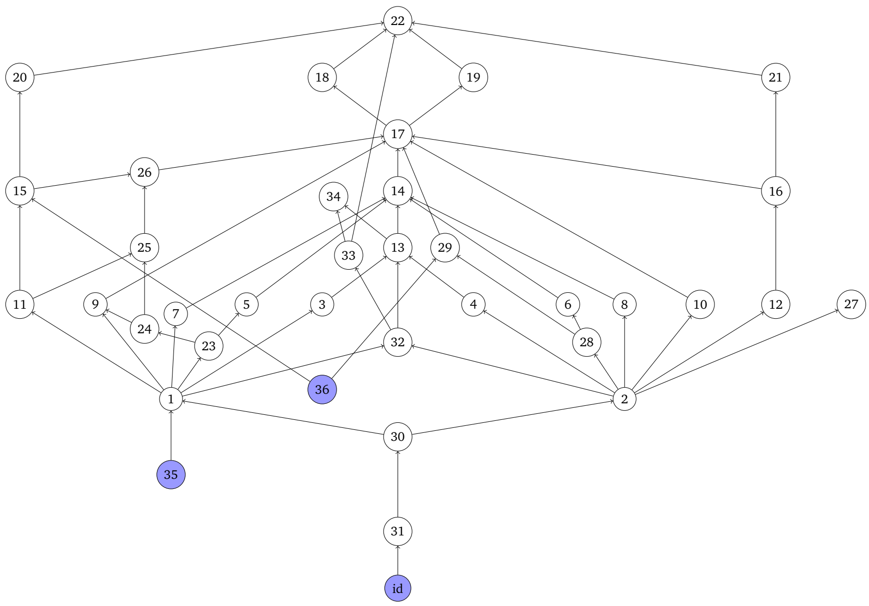


Figure 4.10: Hasse diagram for the upper approximation operators in Table 4.4, where we have extended the Hasse diagram presented in Figure 4.8 with pairs 35 and 36.

properties (D), (INC), (SM), (IU), (ID), (LU), (UE) and (A). By convention, we have considered each pair of Table 4.4 as a pair of dual approximation operators, hence, all pairs 1 – 36 satisfy the property (D). In addition, by the discussion in Section 4.6, all pairs satisfy the property (INC), except the pairs 35 and 36.

First, we study the properties of the pairs of element-based approximation operators, i.e., the pairs 1 – 22 and 31 – 34. We have the following proposition:

Proposition 4.7.1. Let N be a neighborhood operator on U. For the element-based pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ the following holds:

(a) the pair satisfies (D), (SM), (IU) and (UE),

(b) the pair satisfies (INC) if and only if N is reflexive,

(c) the pair satisfies (ID) if and only if N is transitive,

(d) if N is symmetric, then the pair satisfies (LU) if and only if N is transitive,

(e) the pair satisfies (A) if and only if N is symmetric.

Proof. Proofs of the properties can be found in Appendix B. □

Note that for the properties (INC), (ID), (LU) and (A) we provide necessary and sufficient conditions on the neighborhood operator N. It is clear that the properties of $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ follow immediately from the properties of the neighborhood operator N. Recall that the neighborhood operators of groups g, h, j, m, n, o, p and q are symmetric, and the neighborhood operators of groups a, a^{-1}, b, b^{-1}, d, d^{-1}, f, f^{-1}, i, i^{-1}, l, l^{-1}, n and q are transitive. Hence, we obtain the following results:

- the pairs 1 – 22 and 31 – 34 satisfy (SM), (IU) and (UE),
- the pairs 1 – 4, 7, 8, 11, 12, 15, 16, 20, 21, 31 and 34 satisfy (ID),
- the pairs 31 and 34 satisfy (LU),
- the pairs 13, 14, 17, 22 and 31 – 34 satisfy (A).

Note that in [141] the properties (SM), (IU), (ID), (UE) and (A) were discussed for the pairs 1, 2, 5, 6, 11, 13, 14, 17 and 22. However, according to [141], the pair $(\underline{\text{apr}}_f, \overline{\text{apr}}_f)$ does not satisfy (ID), which is incorrect.

Next we consider the pairs 23 – 29. In [141], the properties (SM), (IU), (ID), (UE) and (A) were discussed:

- the property (SM) is satisfied by the pairs 23 - 27,
- the property (IU) is satisfied by none of these pairs,
- the property (ID) is satisfied by the pairs 23 – 26, 28 and 29,
- the property (UE) is satisfied by all of these pairs,
- the property (A) is satisfied by none of these pairs.

To study whether the pairs 23 – 29 satisfy (LU), consider the following two examples.

Example 4.7.2. Let $U = \{1,2\}$ and $\mathbb{C} = \{1,12\}$, then

- $\overline{\text{apr}}'_{\mathbb{C}}(\{2\}) = \{2\} \not\subseteq \emptyset = \underline{\text{apr}}'_{\mathbb{C}}(\overline{\text{apr}}'_{\mathbb{C}}(\{2\}))$,
- $\overline{\text{apr}}'_{\mathbb{C}_\cap}(\{2\}) = \{2\} \not\subseteq \emptyset = \underline{\text{apr}}'_{\mathbb{C}_\cap}(\overline{\text{apr}}'_{\mathbb{C}_\cap}(\{2\}))$,
- $\underline{\text{apr}}_{S_\cap}(\{2\}) = \{2\} \not\supseteq \{1,2\} = \overline{\text{apr}}_{S_\cap}(\underline{\text{apr}}_{S_\cap}(\{2\}))$,
- $H_1^{\mathbb{C}}(\{1\}) = \{1\} \not\subseteq \emptyset = L_1^{\mathbb{C}}(H_1^{\mathbb{C}}(\{1\}))$,
- $H_4^{\mathbb{C}}(\{1\}) = \{1\} \not\subseteq \emptyset = L_4^{\mathbb{C}}(H_4^{\mathbb{C}}(\{1\}))$.

Example 4.7.3. Let $U = \{1,2,3\}$ and $\mathbb{C} = \{12,13\}$, then

- $\overline{\text{apr}}'_{\mathbb{C}_2}(\{2\}) = \{2\} \not\subseteq \emptyset = \underline{\text{apr}}'_{\mathbb{C}_2}(\overline{\text{apr}}'_{\mathbb{C}_2}(\{2\}))$,
- $\overline{\text{apr}}'_{\mathbb{C}_4}(\{2\}) = \{2\} \not\subseteq \emptyset = \underline{\text{apr}}'_{\mathbb{C}_4}(\overline{\text{apr}}'_{\mathbb{C}_4}(\{2\}))$.

Hence, we conclude that none of the pairs 23 – 29 satisfies (LU).

We continue with the study of the properties of the pair (s^-, s^+). From Example 4.5.5, we conclude that the pair does not satisfy (IU). We have the following proposition:

Proposition 4.7.4. Let $(U, \mathbb{C})$ be a covering approximation space, then the pair (s^-, s^+) satisfies the properties (SM), (ID) and (UE).

Proof. By definition, we immediately obtain that (SM) and (UE) are satisfied by the pair (s^-, s^+). For (ID), we prove that $s^+(s^+(X)) \subseteq s^+(X)$ for all $X \subseteq U$, then $s^-(X) \subseteq s^-(s^-(X))$ follows by duality. Let $X \subseteq U$ and $x \in s^+(s^+(X))$, then there exist $y, z \in s^+(X)$ such that $y \in N_1^{\mathbb{C}}(x)$ and $z \in (N_1^{\mathbb{C}})^{-1}(x)$. Since $y \in s^+(X)$, it holds that $N_1^{\mathbb{C}}(y) \cap X \neq \emptyset$, hence, by the transitivity of $N_1^{\mathbb{C}}$ we have that $N_1^{\mathbb{C}}(x) \cap X \neq \emptyset$. On the other hand, as $z \in s^+(X)$, it holds that $(N_1^{\mathbb{C}})^{-1}(z) \cap X \neq \emptyset$, hence, by the transitivity of $(N_1^{\mathbb{C}})^{-1}$ we have that $(N_1^{\mathbb{C}})^{-1}(x) \cap X \neq \emptyset$. We conclude that $x \in s^+(X)$. □

As illustrated in the following example, the pair (s^-, s^+) does not satisfy (LU) and (A).

Example 4.7.5. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{1, 12, 123\}$, then $s^+(\{2\}) = \{2\}$ and $s^-(s^+(\{2\})) = \emptyset$, hence, (s^-, s^+) does not satisfy (LU). On the other hand, we have that $s^+(\{2\}) \subseteq \{2\}$, but $\{2\} \not\subseteq \emptyset = s^-(\{2\})$, hence, (s^-, s^+) does not satisfy (A).

To study the properties of pairs 35 and 36, note that we obtain the following results from [143]: the pair $(\underline{C}_3, \overline{C}_3)$ satisfies the properties (SM) and (UE), but it does not satisfy the properties (IU), (ID) and (LU) and the pair $(\underline{C}^{\%}, \overline{C}^{\%})$ satisfies the properties (SM), (ID) and (UE), but it does not satisfy the properties (IU) and (LU). We determine now by the following two examples that pairs 35 and 36 do not satisfy property (A).

Example 4.7.6. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{1, 3, 13, 24, 34, 14, 234\}$, then $\overline{C}_3(\{2\}) = \emptyset \subseteq \{1, 2\}$, but $\{2\} \not\subseteq \{1\} = \underline{C}_3(\{1, 2\})$.

Example 4.7.7. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{12, 23\}$, then $\overline{C}^{\%}(\{2, 3\}) = \emptyset \subseteq \{3\}$, but $\{2, 3\} \not\subseteq \{3\} = \underline{C}^{\%}(\{3\})$.

Finally, as we now longer work with a binary relation R, the property (RM) is not applicable. To this aim, we introduce a new property indicating monotonicity, now related with coverings. Given a universe U, we define the following partial order relation $\sqsubseteq$ on the set of coverings of U, inspired by the partial relation on partitions of the universe [198]: let $\mathbb{C}$ and $\mathbb{C}'$ be coverings of the universe U, then

$$\mathbb{C} \sqsubseteq \mathbb{C}' \Leftrightarrow (\forall K \in \mathbb{C})(\exists K' \in \mathbb{C}')(K \subseteq K'). \tag{4.4}$$

Given two partitions Π and Π' on U such that $\Pi \sqsubseteq \Pi'$, i.e., Π is *finer* than Π' or Π' is *coarser* than Π, then for Pawlak's rough set model it holds that

$$\underline{\text{apr}}_{\Pi'} \leq \underline{\text{apr}}_{\Pi} \text{ and } \overline{\text{apr}}_{\Pi} \leq \overline{\text{apr}}_{\Pi'},$$

i.e., the pair $(\underline{\text{apr}}_{\Pi}, \overline{\text{apr}}_{\Pi})$ provides more accurate approximations than the pair $(\underline{\text{apr}}_{\Pi'}, \overline{\text{apr}}_{\Pi'})$. When considering coverings instead of partitions, we want to study whether a finer covering provides more accurate approximations. More formally, let $(U, \mathbb{C})$ and $(U, \mathbb{C}')$ be two covering approximation spaces such that $\mathbb{C} \sqsubseteq \mathbb{C}'$. Let $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ be a pair of dual approximation operators in $(U, \mathbb{C})$ and let $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ be the same pair of dual approximation operators in $(U, \mathbb{C}')$, then we want to determine whether $\underline{\text{apr}}_2 \leq \underline{\text{apr}}_1$ and $\overline{\text{apr}}_1 \leq \overline{\text{apr}}_2$ holds. We will call this property *covering monotonicity* and we will denote this as follows:

$$\forall \mathbb{C}, \mathbb{C}' \text{ covering of } U: \mathbb{C} \sqsubseteq \mathbb{C}' \Rightarrow (\underline{\text{apr}}_2, \overline{\text{apr}}_2) \leq (\underline{\text{apr}}_1, \overline{\text{apr}}_1), \tag{CM}$$

with $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ and $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ as described above. We will discuss the property (CM) for the pairs 1 – 36.

Let $(U, \mathbb{C})$ and $(U, \mathbb{C}')$ be two covering approximation spaces such that $\mathbb{C} \sqsubseteq \mathbb{C}'$. Moreover, let $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ be a pair of dual approximation operators in $(U, \mathbb{C})$ and let $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ be the same pair of dual approximation operators in $(U, \mathbb{C}')$, then we want to determine whether $\underline{\text{apr}}_2 \leq \underline{\text{apr}}_1$ and $\overline{\text{apr}}_1 \leq \overline{\text{apr}}_2$ holds. By duality, we can limit ourselves to the comparability of the upper approximation operators.

First, we prove that the property (CM) holds for pair 17:

Proposition 4.7.8. Let $(U, \mathbb{C})$ and $(U, \mathbb{C}')$ be two covering approximation spaces such that $\mathbb{C} \sqsubseteq \mathbb{C}'$, then pair 17 satisfies property (CM).

Proof. Let $X \subseteq U$, then we prove that $\overline{\text{apr}}''_{\mathbb{C}} \leq \overline{\text{apr}}''_{\mathbb{C}'}$. Let $x \in \overline{\text{apr}}''_{\mathbb{C}}(X)$, then there exists $K \in \mathbb{C}$ such that $x \in K$ and $K \cap X \neq \emptyset$. Since $\mathbb{C} \sqsubseteq \mathbb{C}'$, there exists $K' \in \mathbb{C}'$ such that $K \subseteq K'$. Thus, $x \in K'$ and $K' \cap X \neq \emptyset$, hence, $x \in \overline{\text{apr}}''_{\mathbb{C}'}(X)$. □

Moreover, we have the following property for $\sqsubseteq$:

Proposition 4.7.9. Let $(U, \mathbb{C})$ and $(U, \mathbb{C}')$ be two covering approximation spaces such that $\mathbb{C} \sqsubseteq \mathbb{C}'$, then $\mathbb{C}_4 \sqsubseteq (\mathbb{C}')_4$.

Proof. Let $\mathbb{C} \sqsubseteq \mathbb{C}'$ and let $K \in \mathbb{C}_4$, then there exists $x \in K$ such that $K = \bigcup \mathscr{C}(\mathbb{C}, x)$. Since $\mathbb{C} \sqsubseteq \mathbb{C}'$, it holds for each $L \in \mathscr{C}(\mathbb{C}, x)$ that there exists $L' \in \mathbb{C}'$ such that $L \subseteq L'$. As $L' \in \bigcup \mathscr{C}(\mathbb{C}', x)$, we derive that $K \subseteq \bigcup \mathscr{C}(\mathbb{C}', x)$. Hence, there exists $K' \in \mathbb{C}'$, namely $K' = \bigcup \mathscr{C}(\mathbb{C}', x)$, such that $K \subseteq K'$. □

From the two previous propositions, we obtain that pair 22 satisfies the property (CM):

Corollary 4.7.10. Let $(U, \mathbb{C})$ and $(U, \mathbb{C}')$ be two covering approximation spaces such that $\mathbb{C} \sqsubseteq \mathbb{C}'$, then pair 22 satisfies property (CM).

Pairs 17 and 22 are the only pairs which satisfy the property (CM). We provide counterexamples for the other pairs in the following three examples.

Example 4.7.11. Consider $U = \{1,2,3,4\}$ with coverings $\mathbb{C} = \{12, 34\}$ and $\mathbb{C}' = \{12, 34, 14, 23\}$, then $\mathbb{C} \sqsubseteq \mathbb{C}'$. Let $\overline{\text{apr}}_1$ be the upper approximation in $(U, \mathbb{C})$ and $\overline{\text{apr}}_2$ be the same upper approximation in $(U, \mathbb{C}')$.

(a) For pairs 1 – 4, 7, 8, 11 – 13, 15, 16, 20, 21, 23 – 27, 30 – 36, it holds that $\overline{\text{apr}}_1(\{1\}) = \{1,2\} \not\subseteq \{1\} = \overline{\text{apr}}_2(\{1\})$.

(b) For pairs 28 and 29, it holds that

$$\overline{\text{apr}}_1(\{1,4\}) = \{1,2,3,4\} \not\subseteq \{1,4\} = \overline{\text{apr}}_2(\{1,4\}).$$

Example 4.7.12. Consider $U = \{1,2,3,4\}$ with coverings $\mathbb{C} = \{12, 34\}$ and $\mathbb{C}' = \{1, 2, 12, 34\}$, then $\mathbb{C} \sqsubseteq \mathbb{C}'$. Let $\overline{\text{apr}}_1$ be the upper approximation in $(U, \mathbb{C})$ and $\overline{\text{apr}}_2$ be the same upper approximation in $(U, \mathbb{C}')$. For pairs 5, 6, 9, 10 and 14 it holds that $\overline{\text{apr}}_1(\{1\}) = \{1,2\} \not\subseteq \{1\} = \overline{\text{apr}}_2(\{1\})$.

Example 4.7.13. Consider $U = \{1,2,3,4\}$ with coverings $\mathbb{C} = \{12,13,24\}$ and $\mathbb{C}' = \{123,234\}$, then $\mathbb{C} \sqsubseteq \mathbb{C}'$. Let $\overline{\text{apr}}_1$ be the upper approximation in $(U,\mathbb{C})$ and $\overline{\text{apr}}_2$ be the same upper approximation in $(U,\mathbb{C}')$.

(a) For pair 18: $\overline{\text{apr}}_1(\{4\}) = \{1,2,4\} \nsubseteq \{2,3,4\} = \overline{\text{apr}}_2(\{4\})$.

(b) For pair 19: $\overline{\text{apr}}_1(\{1\}) = \{1,2,3,4\} \nsubseteq \{1,2,3\} = \overline{\text{apr}}_2(\{1\})$.

In Table 4.5, we present an overview of the properties and the pairs which satisfies them. For applications such as feature selection, it is advisable that the considered approximation operators satisfy (INC) and (SM). Therefore, the pairs 28, 29, 35 and 36 are less interesting from a practical point of view. On the other hand, pairs 17 and 22 are the only ones which satisfy the property (CM). Although the accuracy provided by these approximation operators is rather low, pairs 17 and 22 contain useful approximation operators from a theoretical point of view. Moreover, pairs 31 and 34 satisfy the properties (INC), (ID) and (LU), hence, for these approximation operators we obtain that:

$$\forall X \in \mathscr{P}(U)\colon \underline{\text{apr}}(\underline{\text{apr}}(X)) = \overline{\text{apr}}(\underline{\text{apr}}(X)) = \underline{\text{apr}}(X),$$
$$\forall X \in \mathscr{P}(U)\colon \underline{\text{apr}}(\overline{\text{apr}}(X)) = \overline{\text{apr}}(\overline{\text{apr}}(X)) = \overline{\text{apr}}(X).$$

4.8 Conclusions and future work

In this chapter, we have constructed a unified framework of dual covering-based approximation operators. We have discussed equalities and partial order relations between the different approximation operators. The results of this research can be found in Table 4.4 and Figure 4.10. By duality, we conclude that the groups 31, 35 and 36 provide the most accurate approximations. In addition, we have discussed properties for all pairs of dual approximation operators and the conclusions of this study are presented in Table 4.5. From this research, we conclude that pairs 35 and 36 are not suitable for applications, since they do not satisfy (INC). Choosing the most suitable covering-based approximation operators in an application will always

Table 4.5: Overview of properties for the covering-based rough set models presented in Table 4.4

Property	Satisfied by the following pairs:
(D)	all pairs
(INC)	all pairs except pairs 35 and 36
(SM)	all pairs except pairs 28 and 29
(IU)	pairs 1 – 22 and 31 – 34
(ID)	pairs 1 – 4, 7, 8, 11, 12, 15, 16, 20, 21, 23 – 26, 28 – 31, 34 and 36
(LU)	pairs 31 and 34
(UE)	all pairs
(A)	pairs 13, 14, 17, 22 and 31 – 34
(CM)	pairs 17 and 22

be a consideration between accuracy on the one hand, and different properties on the other hand and will depend on the goal of the application.

The framework of Restrepo et al. [140], which was the starting point of this research, is extended with 20 different pairs of dual covering-based approximation operators, of which 13 pairs consist of new approximation operators. In Table 4.6, we give an overview of which pairs of approximation operators were already discussed in literature, and which are new. Specifically, we refer to the papers of Restrepo et al. [140, 141], who based their research on work of Yao and Yao [189] and Yang and Li [177], Zhao [197] and Samanta and Chakraborty [142, 143]. From Table 4.6 we see that pairs 3, 4, 7 – 10, 12, 15, 16 and 18 – 21 evoke new dual pairs of approximation operators.

Future work directions include the following:

- Study of the upper approximation operators $H_i^{\mathbb{C}_j}$, with $i \in \{1,4,7\}$ and $\mathbb{C}_j \in \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$, i.e., one of the derived coverings obtained from an original covering $\mathbb{C}$, and their dual lower approximation operator $L_i^{\mathbb{C}_j}$.
- Study of the subsystem-based approximation operators when a derived

covering $\mathbb{C}_j \in \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$ is considered.

- Study of the approximation operators $(\underline{C}^{\%}, \overline{C}^{\%})$ when the union is considered for $H^{\%}$ instead of the intersection (see Eqs. (4.1)). In this case, Remark 4.6.5 would be unnecessary and the results of Proposition 4.6.7 would still hold. In addition, the upper approximation operator would satisfy the inclusion property in contrast to the upper approximation operator presented in [99]. Moreover, we could study the auxiliary operator $H^{\%}$ with other neighborhood operators presented in Table 4.3.

- Study of other theoretical properties. For example, an adaptation of the (CM) property in which we do not consider random coverings, but the coverings $\{\mathbb{C}_A \mid A \subseteq C\}$ defined in Section 3.3.4, with C a set of conditional attributes in a decision table.

- The application of covering-based rough approximation operators in machine learning techniques such as feature and instance selection.

Table 4.6: Overview of literature

Pair	[141]	[197]	[143]	Pair	[141]	[197]	[143]
1	x	x	x	19			
2	x	x	x	20			
3				21			
4				22	x		
5	x			23	x		x
6	x		x	24	x		
7				25	x		
8				26	x		x
9				27	x		
10				28	x		x
11	x			29	x		x
12				30		x	
13	x		x	31		x	x
14	x			32		x	
15				33		x	
16				34		x	
17	x		x	35			x
18				36			x

CHAPTER 5

Preliminary notions of fuzzy set theory

Applications of rough set theory are widespread and are especially prominent in data analysis [94, 95] and more specifically in feature selection and classification [154]. However, since the traditional rough set model of Pawlak [128] is designed to process qualitative (discrete) data, it faces important limitations when dealing with real-valued data sets [81]. Fuzzy set theory proposed in 1965 by Zadeh [193] is very useful to overcome these limitations, as it can deal effectively with vague concepts and graded indiscernibility.

In the following, we will discuss some preliminary notions regarding fuzzy set theory. First, we discuss fuzzy logical connectives in Section 5.1 and fuzzy set theory in Section 5.2. In Section 5.3, we recall some aggregation operators. To end, we will discuss the technique of representation by levels, introduced by Sánchez et al. [144] in Section 5.4.

5.1 Fuzzy logical connectives

We recall some important fuzzy logical connectives on the unit interval $[0,1]$. First, we discuss conjunctors, disjunctors and negators which are fuzzy extensions of the Boolean conjunction $\wedge$, disjunction $\vee$ and negation $\neg$.

A *conjunctor* is a mapping $\mathscr{C}: [0,1]\times[0,1]\rightarrow[0,1]$ which is increasing in both arguments and which satisfies the boundary conditions

$$\begin{aligned}&\mathscr{C}(0,0)=\mathscr{C}(0,1)=\mathscr{C}(1,0)=0,\\&\mathscr{C}(1,1)=1.\end{aligned}$$

It is called a *border conjunctor* if for all $a\in[0,1]$ it holds that $\mathscr{C}(1,a)=a$. A commutative and associative border conjunctor is called a *triangular norm* or *t-norm* and is denoted by $\mathscr{T}$. We provide some examples of t-norms:

(a) the standard minimum operator $\mathscr{T}_M$ defined by $\forall a,b\in[0,1]$:

$$\mathscr{T}_M(a,b)=\min(a,b),$$

(b) the product operator $\mathscr{T}_P$ defined by $\forall a,b\in[0,1]$: $\mathscr{T}_P(a,b)=a\cdot b$,

(c) the bold intersection or Łukasiewicz t-norm $\mathscr{T}_L$ defined by $\forall a,b\in[0,1]$:

$$\mathscr{T}_L(a,b)=\max(0,a+b-1),$$

(d) the cosine t-norm $\mathscr{T}_{\cos}$ defined by $\forall a,b\in[0,1]$:

$$\mathscr{T}_{\cos}(a,b)=\max\left(0,ab-\sqrt{(1-a^2)(1-b^2)}\right),$$

(e) the drastic t-norm $\mathscr{T}_D$ defined by $\forall a,b\in[0,1]$:

$$\mathscr{T}_D(a,b)=\begin{cases}b & a=1\\ a & b=1\\ 0 & \text{otherwise,}\end{cases}$$

(f) the nilpotent minimum t-norm $\mathscr{T}_{nM}$ defined by $\forall a,b \in [0,1]$:

$$\mathscr{T}_{nM}(a,b) = \begin{cases} \min(a,b) & a+b>1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that for every t-norm $\mathscr{T}$ it holds that

$$\forall a,b \in [0,1]\colon \mathscr{T}_M(a,b) \geq \mathscr{T}(a,b) \geq \mathscr{T}_D(a,b).$$

It is often assumed that the t-norm $\mathscr{T}$ is left-continuous, i.e., it is left-continuous in both parameters. Since $\mathscr{T}$ is commutative, it is sufficient to assume that $\mathscr{T}$ is left-continuous in the first parameter, i.e.,

$$\begin{aligned} &(\forall a,b \in [0,1])(\forall \epsilon > 0)(\exists \delta > 0)(\forall c \in [0,1])\colon \\ &\qquad a-\delta < c < a \Rightarrow |\mathscr{T}(c,b) - \mathscr{T}(a,b)| < \epsilon. \end{aligned} \tag{5.1}$$

Furthermore, a t-norm $\mathscr{T}$ is left-continuous if and only if it is complete-distributive with respect to the supremum, i.e., for every family $(a_i)_{i\in I}$ in $[0,1]$ it holds that

$$\forall b \in [0,1]\colon \mathscr{T}\left(\sup_{i\in I} a_i, b\right) = \sup_{i\in I} \mathscr{T}(a_i,b). \tag{5.2}$$

A *disjunctor* is a mapping $\mathscr{D}\colon [0,1]\times[0,1] \to [0,1]$ which is increasing in both arguments and which satisfies the boundary conditions

$$\begin{aligned} &\mathscr{D}(1,1) = \mathscr{D}(0,1) = \mathscr{D}(1,0) = 1, \\ &\mathscr{D}(0,0) = 0. \end{aligned}$$

It is called a *border disjunctor* if for all $a \in [0,1]$ it holds that $\mathscr{D}(0,a) = a$. A commutative and associative border disjunctor is called a *triangular conorm* or *t-conorm* and is denoted by $\mathscr{S}$. We provide some examples of t-conorms:

(a) the standard maximum operator $\mathscr{S}_M$ defined by $\forall a,b \in [0,1]$:

$$\mathscr{S}_M(a,b) = \max(a,b),$$

(b) the probabilistic sum $\mathscr{S}_P$ defined by $\forall a,b \in [0,1]$:

$$\mathscr{S}_P(a,b) = a + b - a\cdot b,$$

(c) the bounded sum or Łukasiewicz t-conorm $\mathscr{S}_L$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{S}_L(a,b) = \min(1, a+b),$$

(d) the cosine t-conorm $\mathscr{S}_{\cos}$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{S}_{\cos}(a,b) = \min\left(1, a+b-ab+\sqrt{(2a-a^2)(2b-2b^2)}\right),$$

(e) the drastic t-conorm $\mathscr{S}_D$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{S}_D(a,b) = \begin{cases} b & a=0 \\ a & b=0 \\ 1 & \text{otherwise,} \end{cases}$$

(f) the nilpotent maximum t-conorm $\mathscr{S}_{nM}$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{S}_{nM}(a,b) = \begin{cases} \max(a,b) & a+b<1 \\ 1 & \text{otherwise.} \end{cases}$$

Note that for every t-conorm $\mathscr{S}$ it holds that

$$\forall a, b \in [0,1]\colon \mathscr{S}_M(a,b) \leq \mathscr{S}(a,b) \leq \mathscr{S}_D(a,b).$$

A *negator* is a decreasing mapping $\mathscr{N}\colon [0,1] \to [0,1]$ which satisfies $\mathscr{N}(0) = 1$ and $\mathscr{N}(1) = 0$. A negator is called *involutive* if for all $a \in [0,1]$ it holds that $\mathscr{N}(\mathscr{N}(a)) = a$. Some widespread negators are the standard negator $\mathscr{N}_S$ defined by $\forall a \in [0,1]\colon \mathscr{N}_S(a) = 1-a$ and the Gödel negator $\mathscr{N}_G$ defined by $\forall a \in [0,1]$:

$$\mathscr{N}_G(a) = \begin{cases} 1 & a=0 \\ 0 & a>0. \end{cases}$$

Given an involutive negator $\mathscr{N}$, a conjunctor $\mathscr{C}$ and a disjunctor $\mathscr{D}$, we can define the $\mathscr{N}$-dual of $\mathscr{C}$ and $\mathscr{D}$. The *$\mathscr{N}$-dual of the conjunctor $\mathscr{C}$* is a disjunctor $\mathscr{D}_{\mathscr{C},\mathscr{N}}$ defined by

$$\forall a, b \in [0,1] : \mathscr{D}_{\mathscr{C},\mathscr{N}}(a,b) = \mathscr{N}(\mathscr{C}(\mathscr{N}(a), \mathscr{N}(b)))$$

and the *$\mathcal{N}$-dual of the disjunctor $\mathcal{D}$* is a conjunctor $\mathcal{C}_{\mathcal{D},\mathcal{N}}$ defined by

$$\forall a, b \in [0,1]\colon \mathcal{C}_{\mathcal{D},\mathcal{N}}(a,b) = \mathcal{N}(\mathcal{D}(\mathcal{N}(a), \mathcal{N}(b))).$$

It can be verified that the $\mathcal{N}$-dual of a t-norm is a t-conorm and vice versa. For example, all the t-conorms mentioned above are the $\mathcal{N}_S$-dual of the respective t-norms.

Besides conjunctors, disjunctors and negators, we recall the notion of implicators. They extend the Boolean implication $\Rightarrow$ to the fuzzy setting.

An *implicator* is a mapping $\mathcal{I}\colon [0,1] \times [0,1] \to [0,1]$ which is decreasing in the first and increasing in the second argument and which satisfies the boundary conditions $\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1$ and $\mathcal{I}(1,0) = 0$. It is called a *border implicator* if for all $a \in [0,1]$ it holds that $\mathcal{I}(1,a) = a$. It satisfies the *weak confinement principle* if $\forall a, b \in [0,1]\colon a \leq b \Rightarrow \mathcal{I}(a,b) = 1$. An implicator which satisfies the weak confinement principle is called a *WCP-implicator*.

We can define negators based on implicators. Let $\mathcal{I}$ be an implicator. The *induced negator* of $\mathcal{I}$ is the negator $\mathcal{N}_{\mathcal{I}}$ defined by

$$\forall a \in [0,1]\colon \mathcal{N}_{\mathcal{I}}(a) = \mathcal{I}(a,0).$$

There are two important classes of implicators: S-implicators based on disjunctors and negators and R-implicators based on border conjunctors.

- Let $\mathcal{D}$ be a disjunctor and $\mathcal{N}$ be a negator. The *S-implicator* $\mathcal{I}_{\mathcal{D},\mathcal{N}}$ based on the disjunctor $\mathcal{D}$ and the negator $\mathcal{N}$ is defined by

$$\forall a, b \in [0,1]\colon \mathcal{I}_{\mathcal{D},\mathcal{N}}(a,b) = \mathcal{D}(\mathcal{N}(a), b).$$

- Let $\mathcal{C}$ be a border conjunctor. The *residual implicator* or *R-implicator* $\mathcal{I}_{\mathcal{C}}$ based on the border conjunctor $\mathcal{C}$ is defined by

$$\forall a, b \in [0,1]\colon \mathcal{I}_{\mathcal{C}}(a,b) = \sup\{c \in [0,1] \mid \mathcal{C}(a,c) \leq b\}.$$

Both S- and R-implicators are border implicators. R-implicators are also WCP-implicators. A left-continuous t-norm $\mathscr{T}$ and its R-implicator $\mathscr{I}$ satisfy the residuation principle:

$$\forall a, b, c \in [0,1]\colon \mathscr{T}(a,b) \leq c \Leftrightarrow a \leq \mathscr{I}(b,c). \tag{5.3}$$

A special class of R-implicators are *IMTL-implicators*, where IMTL stands for 'Involutive Monoidal T-norm based Logic' [43, 49]: these R-implicators are based on a left-continuous t-norm and have an involutive induced negator. A left-continuous t-norm of which the R-implicator is an IMTL-implicator, is called an *IMTL-t-norm*. Note that every IMTL-implicator is also a S-implicator. We provide some examples of implicators:

(a) the Kleene-Dienes implicator $\mathscr{I}_{KD}$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{I}_{KD}(a,b) = \max(1-a, b)$$

is a S-implicator based on the maximum and the standard negator,

(b) the Gödel implicator $\mathscr{I}_G$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{I}_G(a,b) = \begin{cases} 1 & a \leq b \\ b & a > b \end{cases}$$

is an R-implicator based on the minimum,

(c) the Łukasiewicz implicator $\mathscr{I}_L$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{I}_L(a,b) = \min(1, 1-a+b)$$

is an IMTL-implicator based on the Łukasiewicz t-norm,

(d) the nilpotent minimum implicator $\mathscr{I}_{nM}$ defined by $\forall a, b \in [0,1]$:

$$\mathscr{I}_{nM}(a,b) = \begin{cases} 1 & a \leq b \\ \max(1-a, b) & a > b \end{cases}$$

is an IMTL-implicator based on the nilpotent minimum.

The standard negator $\mathcal{N}_S$ is the induced negator of the implicators $\mathcal{I}_{KD}$, $\mathcal{I}_L$ and $\mathcal{I}_{nM}$. The induced negator of the Gödel implicator $\mathcal{I}_G$ is the Gödel negator $\mathcal{N}_G$.

Finally, we recall that we can construct conjunctors based on implicators and involutive negators. Given an involutive negator $\mathcal{N}$ and an implicator $\mathcal{I}$, the *induced conjunctor of the implicator $\mathcal{I}$ and the negator $\mathcal{N}$* is the conjunctor $\mathcal{C}_{\mathcal{I},\mathcal{N}}$ defined by

$$\forall a, b \in [0,1]\colon \mathcal{C}_{\mathcal{I},\mathcal{N}}(a,b) = \mathcal{N}(\mathcal{I}(a, \mathcal{N}(b))).$$

Note that $\mathcal{C}_{\mathcal{I},\mathcal{N}}$ is not necessarily a t-norm.

5.2 Fuzzy set theory

Next, we recall basic notions on fuzzy set theory [193].

A *fuzzy set* X in a non-empty universe U is a mapping $X: U \to [0,1]$, i.e., each object $x \in U$ is associated with a *membership degree* $X(x) \in [0,1]$. The collection of all fuzzy sets in U is denoted by $\mathcal{F}(U)$. If U is finite, the *cardinality* of X is defined by

$$|X| = \sum_{x \in U} X(x).$$

Note that for a finite universe U, we often use the following notation for a fuzzy set X of U: let $U = \{x_1, x_2, \ldots, x_n\}$, then we denote X by

$$X = X(x_1)/x_1 + X(x_2)/x_2 + \ldots + X(x_n)/x_n.$$

The *support* of X is the crisp set $\text{supp}(X) = \{x \in U \mid X(x) > 0\}$. Given $\alpha \in [0,1]$, the *α-level set* X_α of X in U is a crisp set in U such that $x \in X_\alpha$ if and only if $X(x) \geq \alpha$.

We describe some special fuzzy sets. Given $\alpha \in [0,1]$, the *constant (fuzzy) set* $\hat{\alpha}$ in U is defined by

$$\forall x \in U\colon \hat{\alpha}(x) = \alpha.$$

In the crisp case, the only constant sets in U are $\hat{0} = \emptyset$ and $\hat{1} = U$. Moreover, for $x \in U$, the fuzzy set 1_x is defined by $1_x(x) = 1$ and $1_x(y) = 0$ for all $y \in U \setminus \{x\}$.

For two fuzzy sets X and Y of U, we say that X is included in Y, denoted by $X \subseteq Y$, if and only if $\forall x \in U: X(x) \leq Y(x)$.

In the crisp case, the complement is an operator on $\mathscr{P}(U)$ and the union and the intersection are operators from $\mathscr{P}(U) \times \mathscr{P}(U)$ to $\mathscr{P}(U)$. We can extend these notions to the fuzzy setting. Let $X \in \mathscr{F}(U)$ and $\mathscr{N}$ a negator, then the *$\mathscr{N}$-complement $X^{\mathscr{N}}$ of X* is given by

$$\forall x \in U : X^{\mathscr{N}}(x) = \mathscr{N}(X(x)). \tag{5.4}$$

Moreover, let $X, Y \in \mathscr{F}(U)$ and $\mathscr{C}$ a conjunctor, then the *$\mathscr{C}$-intersection of X and Y* is the fuzzy set $X \cap_{\mathscr{C}} Y$ defined by

$$\forall x \in U : (X \cap_{\mathscr{C}} Y)(x) = \mathscr{C}(X(x), Y(x)), \tag{5.5}$$

and the *$\mathscr{D}$-union of X and Y* is the fuzzy set $X \cup_{\mathscr{D}} Y$ defined by

$$\forall x \in U : (X \cup_{\mathscr{D}} Y)(x) = \mathscr{D}(X(x), Y(x)), \tag{5.6}$$

When the minimum and maximum operator are considered, we write $X \cap Y$ and $X \cup Y$ instead of $X \cap_{\mathscr{T}_M} Y$ and $X \cup_{\mathscr{S}_M} Y$.

Let us also recall fuzzy relations. A *(binary) fuzzy relation R* is a fuzzy set in the Cartesian product $U \times U$, i.e., $R \in \mathscr{F}(U \times U)$. The tuple (U, R) is called a *fuzzy relation approximation space*. The *inverse fuzzy relation R^{-1}* of R is the fuzzy set in $U \times U$ defined by

$$\forall x, y \in U : R^{-1}(x, y) = R(y, x). \tag{5.7}$$

Given a fuzzy relation R and an object $x \in U$, then we can define the *fuzzy set of predecessors of x* denoted by $R^p(x)$ as follows

$$\forall y \in U : (R^p(x))(y) = R(y, x), \tag{5.8}$$

and we can define the *fuzzy set of successors of x* denoted by $R^s(x)$ as follows

$$\forall y \in U : (R^s(x))(y) = R(x, y). \tag{5.9}$$

Furthermore, given a fuzzy relation R, then R can satisfy the following properties:

- R is serial if and only if $\forall x \in U : \sup\limits_{y \in U} R(x, y) = 1$,

- R is strongly serial if and only if $\forall x \in U \exists y \in U: R(x,y) = 1$,
- R is inverse serial if and only if $\forall x \in U: \sup_{y \in U} R(y,x) = 1$,
- R is strongly inverse serial if and only if $\forall x \in U \exists y \in U: R(y,x) = 1$,
- R is reflexive if and only if $\forall x \in U: R(x,x) = 1$,
- R is symmetric if and only if $\forall x, y \in U: R(x,y) = R(y,x)$,
- given a t-norm $\mathcal{T}$, R is $\mathcal{T}$-transitive if and only if

$$\forall x, y, z \in U: \mathcal{T}(R(x,y), R(y,z)) \leq R(x,z),$$

- given a t-norm $\mathcal{T}$, R is $\mathcal{T}$-Euclidean if and only if

$$\forall x, y, z \in U: \mathcal{T}(R(y,x), R(y,z)) \leq R(x,z).$$

A fuzzy relation which is reflexive and symmetric is called a *fuzzy tolerance relation*. If the fuzzy relation is reflexive and $\mathcal{T}$-transitive for a t-norm $\mathcal{T}$, then the relation is called a *fuzzy $\mathcal{T}$-dominance relation* or *fuzzy $\mathcal{T}$-preorder*. If the fuzzy relation R is reflexive, symmetric and $\mathcal{T}$-transitive for a t-norm $\mathcal{T}$, the relation is called a *fuzzy $\mathcal{T}$-similarity relation*. Moreover, every fuzzy relation which is reflexive and $\mathcal{T}$-Euclidean is a fuzzy $\mathcal{T}$-similarity relation [163]. Note that for the t-norm $\mathcal{T}_M$, we write transitive, Euclidean and fuzzy similarity relation instead of $\mathcal{T}_M$-transitive, $\mathcal{T}_M$-Euclidean and fuzzy $\mathcal{T}_M$-similarity relation.

5.3 Aggregation operators

Generally speaking, an *aggregation operator* is an operator which provides one numerical value for a set of numerical values. More formally, the function

$$f: \mathcal{U}^n \to \mathcal{U}$$

is an aggregation operator of order n on a domain $\mathcal{U}$. In this work, we will always use the domain $\mathcal{U} = [0,1]$. In the following, we discuss some aggregation operators.

First, we study aggregation operators based on t-norms and t-conorms. Since t-norms and t-conorms are associative and commutative, we can extend these binary operators to n-ary operators as follows. Let $\mathscr{T}$ be a t-norm, then we can define the i-ary operators $\mathscr{T}^i : [0,1]^i \to [0,1]$ for $i \in \mathbb{N}$, $i \geq 2$ as follows:

$$\begin{aligned} &\forall a_1, a_2 \in [0,1] \colon \mathscr{T}^2(a_1,a_2) = \mathscr{T}(a_1,a_2), \\ &\forall i \in \mathbb{N}, i \geq 3, \forall (a_1,a_2,\ldots,a_{i-1},a_i) \in [0,1]^i \colon \\ &\qquad \mathscr{T}^i(a_1,a_2,\ldots,a_{i-1},a_i) = \mathscr{T}(\mathscr{T}^{i-1}(a_1,a_2,\ldots,a_{i-1}),a_i). \end{aligned} \tag{5.10}$$

Similarly, let $\mathscr{S}$ be a t-conorm, then we can define for $i \in \mathbb{N}$, $i \geq 2$, the i-ary operators $\mathscr{S}^i : [0,1]^i \to [0,1]$ as follows:

$$\begin{aligned} &\forall a_1, a_2 \in [0,1] \colon \mathscr{S}^2(a_1,a_2) = \mathscr{S}(a_1,a_2), \\ &\forall i \in \mathbb{N}, i \geq 3, \forall (a_1,a_2,\ldots,a_{i-1},a_i) \in [0,1]^i \colon \\ &\qquad \mathscr{S}^i(a_1,a_2,\ldots,a_{i-1},a_i) = \mathscr{S}(\mathscr{S}^{i-1}(a_1,a_2,\ldots,a_{i-1}),a_i). \end{aligned} \tag{5.11}$$

Fernández-Salido and Murakami [47] proposed the β-precision quasi t-norm and t-conorm for $\beta \in [0,1]$ as aggregation operators. Let $\mathscr{T}$ be a t-norm, $\mathscr{S}$ be a t-conorm, $\beta \in [0,1]$ and $n \in \mathbb{N}$ with $n \geq 2$, then the *β-precision quasi-t-norm* $\mathscr{T}_\beta$ and the *β-precision quasi-t-conorm* $\mathscr{S}_\beta$ of order n are $[0,1]^n \to [0,1]$ mappings such that for all $(a_1,a_2,\ldots,a_{n-1},a_n)$ in $[0,1]^n$,

$$\mathscr{T}_\beta(a_1,a_2,\ldots,a_{n-1},a_n) = \mathscr{T}^{n-m}(b_1,\ldots,b_{n-m}), \tag{5.12}$$

$$\mathscr{S}_\beta(a_1,a_2,\ldots,a_{n-1},a_n) = \mathscr{S}^{n-p}(c_1,\ldots,c_{n-p}), \tag{5.13}$$

where b_i is the i^{th} greatest element of $(a_1,a_2,\ldots,a_{n-1},a_n)$ and c_i is the i^{th} smallest element of $(a_1,a_2,\ldots,a_{n-1},a_n)$, and

$$\begin{aligned} m &= \max\left\{ i \in \{0,\ldots,n\} \mid i \leq (1-\beta)\sum_{j=1}^{n} a_j \right\}, \\ p &= \max\left\{ i \in \{0,\ldots,n\} \mid i \leq (1-\beta)\sum_{j=1}^{n} (1-a_j) \right\}. \end{aligned}$$

When $\beta = 1$, we obtain the operators $\mathscr{T}^n$ and $\mathscr{S}^n$ for $n \in \mathbb{N} \setminus \{0,1\}$.

Second, we discuss ordered weighted average (OWA) operators, introduced by Yager [175]. Let D be a sequence of n scalar values and let $W = \langle w_1, w_2, \ldots, w_n \rangle$ be a weight vector of length n, henceforth called an *OWA weight vector of length n*, such that for all $i \in \{1, \ldots, n\}$, $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$. Let σ be the permutation on $\{1, \ldots, n\}$ such that $d_{\sigma(i)}$ is the i^{th} largest value of D. The *OWA operator acting on D* yields the value

$$\text{OWA}_W(D) = \sum_{i=1}^{n} w_i d_{\sigma(i)}. \tag{5.14}$$

The OWA operator allows us to consider a wide variety of aggregation strategies. We consider the following examples. Let D be a sequence of length n:

(a) for $W_{\text{min}} = \langle 0, 0, \ldots, 0, 1 \rangle$, the result of $\text{OWA}_{W_{\text{min}}}(D)$ equals the minimum value of D,

(b) for $W_{\text{max}} = \langle 1, 0, \ldots, 0, 0 \rangle$, the result of $\text{OWA}_{W_{\text{max}}}(D)$ equals the maximum value of D,

(c) for $W_{\text{avg}} = \langle \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, \frac{1}{n} \rangle$, the result of $\text{OWA}_{W_{\text{avg}}}(D)$ represents the average of D.

Moreover, given an OWA weight vector W of length n, the *orness degree* and *andness degree* of W are defined by

$$\begin{aligned} \text{orness}(W) &= \frac{1}{n-1} \sum_{i=1}^{n} ((n-i) \cdot w_i), & (5.15) \\ \text{andness}(W) &= 1 - \text{orness}(W). & (5.16) \end{aligned}$$

The orness and andness degree of W represent how similar W is to the maximum and minimum operator, respectively. For instance, it holds that

(a) $\text{orness}(W_{\text{max}}) = 1$ and $\text{andness}(W_{\text{max}}) = 0$,

(b) $\text{orness}(W_{\text{min}}) = 0$ and $\text{andness}(W_{\text{min}}) = 1$,

(c) $\text{orness}(W_{\text{avg}}) = \text{andness}(W_{\text{avg}}) = 0.5$.

5.4 Technique of representation by levels

In 2012, Sánchez et al. [144] introduced a non-nested level-based representation of fuzziness. The idea is to describe a fuzzy concept with crisp representatives, each one being a crisp realization under a certain condition [144]. Different levels of restriction are considered, with using the levels in $[0,1]$, where level 1 is the most restrictive level. Level 0 represents no restriction at all, but it will not be taken into account in the representation. Since humans can only distinguish a finite set of levels, for each fuzzy concept X it is assumed that there exists a finite set of levels $\Lambda_X = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0$ and $m \in \mathbb{N} \setminus \{0\}$[4].

A fuzzy concept X is described by a *representation by levels* (RL) which is a tuple (Λ_X, ρ_X) with Λ_X a finite set of levels and $\rho_X : \Lambda_X \to \mathscr{P}(X)$ a function which associates each level α with a crisp subset of X. The set of crisp representatives Ω_X of (Λ_X, ρ_X) is given by $\Omega_X = \{\rho_X(\alpha) \mid \alpha \in \Lambda_X\}$. Furthermore, the crisp representatives on each level are independent of each other and they are not necessarily nested, i.e., $\alpha > \beta \not\Rightarrow \rho_X(\alpha) \supseteq \rho_X(\beta)$ for X a fuzzy concept. Note that a fuzzy set X can be seen as a special case of RL, in case there are only finitely many different membership degrees: let $\Lambda_X = \{X(x) \mid X(x) > 0\} \cup \{1\}$ and $\rho_X(\alpha) = X_\alpha$ for each $\alpha \in \Lambda_X$. In this case, the crisp representatives are nested.

Although this technique is useful to represent fuzzy information, it is not easy to interpret by humans. Therefore, it is possible to obtain a fuzzy set that summarizes the information given by the RL: let (Λ_X, ρ_X) be an RL associated with a fuzzy concept X, then the *fuzzy summary* $v_X : U \to [0,1]$ is given by

$$\begin{aligned} v_X(x) &= \sum_{\{Y \in \Omega_X \mid x \in Y\}} \left(\sum_{\{\alpha_i \in \Lambda_X \mid Y = \rho_X(\alpha_i)\}} (\alpha_i - \alpha_{i+1}) \right) \\ &= \sum_{\{\alpha_i \in \Lambda_X \mid x \in \rho_X(\alpha_i)\}} (\alpha_i - \alpha_{i+1}), \end{aligned}$$

i.e., we take the summation of the differences $\alpha_i - \alpha_{i+1}$, where x belongs to the crisp representative on level α_i.

[4] It is possible to consider a countable set of levels. However, we will restrict to finite sets of levels.

Considering operations on fuzzy concepts, this technique will allow to perform the associated crisp operations on each level of the RL. Let $f : \mathscr{P}(U)^n \to \mathscr{P}(U)$ be a crisp operation, then f is extended to RLs in the following way: let $(X_1, X_2, \ldots, X_n)$ be fuzzy concepts in U with each X_i represented by $(\Lambda_{X_i}, \rho_{X_i})$, then $f(X_1, X_2, \ldots, X_n)$ is a fuzzy concept in U represented by $(\Lambda_{f(X_1,X_2,\ldots,X_n)}, \rho_{f(X_1,X_2,\ldots,X_n)})$ where

$$\Lambda_{f(X_1,X_2,\ldots,X_n)} = \bigcup_{1 \leq i \leq n} \Lambda_{X_i}$$

and

$$\forall \alpha \in \Lambda_{f(X_1,X_2,\ldots,X_n)} : \rho_{f(X_1,X_2,\ldots,X_n)}(\alpha) = f(\rho_{X_1}(\alpha), \rho_{X_2}(\alpha), \ldots, \rho_{X_n}(\alpha)).$$

Examples of such operations are the union, the intersection or the complement. We have the following proposition:

Proposition 5.4.1. [144] Operations on RLs satisfy all the properties of the Boolean logic.

In other words, all properties using operations of the Boolean logic, e.g., negation, conjunction and disjunction, which hold for a crisp concept will also hold for its fuzzification, when RLs are used. This is the main advantage of non-nested level-based representations.

Finally, we illustrate this technique with following example:

Example 5.4.2. Let $U = \{x, y, z\}$ and consider the fuzzy sets

- $X = 1/x + 0.8/y + 0.5/z$,
- $Y = 0.9/x + 0/y + 0.6/z$.

Let $\mathscr{T}$ be a t-norm and $\mathscr{N}$ a negator, then we want to determine the fuzzy set $X \cap_{\mathscr{T}} Y^{\mathscr{N}}$ by the technique of representations by levels. First, we determine the set of levels: $\Lambda_{(X \cap_{\mathscr{T}} Y^{\mathscr{N}})} = \Lambda_X \cup \Lambda_Y = \{1, 0.9, 0.8, 0.6, 0.5\}$, with Λ_X and Λ_Y the set of non-zero membership degrees of X and Y. For each level $\alpha \in \Lambda_{(X \cap_{\mathscr{T}} Y^{\mathscr{N}})}$, we determine the crisp set $\rho_X(\alpha)$, $\rho_Y(\alpha)$, $\rho_{Y^{\mathscr{N}}}(\alpha)$ and $\rho_{(X \cap_{\mathscr{T}} Y^{\mathscr{N}})}(\alpha)$ in Table 5.1.

Table 5.1: Representation by levels of X, Y, $Y^{\mathscr{N}}$ and $X \cap_{\mathscr{T}} Y^{\mathscr{N}}$

α	$\rho_X(\alpha)$	$\rho_Y(\alpha)$	$\rho_{Y^{\mathscr{N}}}(\alpha)$	$\rho_{(X \cap_{\mathscr{T}} Y^{\mathscr{N}})}(\alpha)$
1	$\{x\}$	$\emptyset$	$\{x,y,z\}$	$\{x\}$
0.9	$\{x\}$	$\{x\}$	$\{y,z\}$	$\emptyset$
0.8	$\{x,y\}$	$\{x\}$	$\{y,z\}$	$\{y\}$
0.6	$\{x,y\}$	$\{x,z\}$	$\{y\}$	$\{y\}$
0.5	$\{x,y,z\}$	$\{x,z\}$	$\{y\}$	$\{y\}$

From Table 5.1 we see that the crisp representatives of $Y^{\mathscr{N}}$ and $X \cap_{\mathscr{T}} Y^{\mathscr{N}}$ are not nested. To determine the fuzzy set $X \cap_{\mathscr{T}} Y^{\mathscr{N}}$, we compute the fuzzy summary in each object of U:

$$\begin{aligned}(X \cap_{\mathscr{T}} Y^{\mathscr{N}})(x) &= (1-0.9) = 0.1,\\ (X \cap_{\mathscr{T}} Y^{\mathscr{N}})(y) &= (0.8-0.6)+(0.6-0.5)+(0.5-0) = 0.8,\\ (X \cap_{\mathscr{T}} Y^{\mathscr{N}})(z) &= 0,\end{aligned}$$

thus, $X \cap_{\mathscr{T}} Y^{\mathscr{N}} = 0.1/x + 0.8/y + 0/z$. Note that this holds for every t-norm $\mathscr{T}$ and every negator $\mathscr{N}$.

CHAPTER 6

Fuzzy neighborhood operators

In machine learning processes, neighborhood operators play an important role as they are generalizations of equivalence classes which were used in the original rough set model of Pawlak. In this chapter, we introduce the notions of a fuzzy covering, the fuzzy neighborhood system of an object, the fuzzy minimal description of an object and the fuzzy maximal description of an object based on a given fuzzy covering. Moreover, we extend the definition of four covering-based neighborhood operators as well as six derived coverings discussed by Yao and Yao [189] to the fuzzy setting. We combine these four fuzzy neighborhood operators and seven fuzzy coverings and prove that only 16 different fuzzy neighborhood operators are obtained when a finite fuzzy covering, a left-continuous t-norm and its R-implicator are considered. In addition, we study the partial order relations between those 16 fuzzy neighborhood operators and a fuzzy neighborhood operator introduced by Ma [107]. Furthermore, we discuss which properties are satisfied by each fuzzy neighborhood operator. To end, we state conclusions.

Note that in the literature, fuzzy neighborhood operators are often used in the context of fuzzy topology, e.g., [86, 93, 103, 105], in order to describe concepts such as open and closed sets, and interior and closure operators. Interior and closure operators are closely related with the concept of approximation operators in data analysis, and here we focus on the concept of a fuzzy neighborhood operator from the perspective of fuzzy rough set theory.

6.1 Fuzzy neighborhood systems based on a fuzzy covering

We first discuss the concept of a fuzzy covering. Different definitions of a fuzzy covering are proposed in [36, 91]. However, we introduce the following one:

Definition 6.1.1. Let U be a universe and let I be a possibly infinite index set. A collection $\mathbb{C} = \{K_i \in \mathscr{F}(U) \mid K_i \neq \emptyset, i \in I\}$ is called a *fuzzy covering*, if for all $x \in U$ there exists $K \in \mathbb{C}$ such that $K(x) = 1$. The tuple $(U, \mathbb{C})$ is called a *fuzzy covering approximation space*.

Note that for infinite coverings this definition guarantees for any $x \in U$ the existence of a set $K \in \mathbb{C}$ to which x fully belongs, which is not the case with the proposals of [36, 91]. Moreover, note that every crisp covering of U is also a fuzzy covering of U, since $\mathscr{P}(U) \subseteq \mathscr{F}(U)$. Therefore, we will use the same notation for a crisp and a fuzzy covering. From now on, we assume that $\mathbb{C}$ represents a fuzzy covering, unless explicitly indicated otherwise.

Given a fuzzy covering $\mathbb{C}$ and an object $x \in U$, we want to describe the fuzzy neighborhood system, the fuzzy minimal description and the fuzzy maximal description of x. We first introduce the following extension of the neighborhood system of $x \in U$, which is the collection of all fuzzy sets in the fuzzy covering in which x has a strict positive membership degree.

Definition 6.1.2. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $x \in U$, then the *fuzzy neighborhood system* of x is given by

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} \mid K(x) > 0\}. \tag{6.1}$$

By definition of a fuzzy covering, there is always a set $K \in \mathbb{C}$ with $K(x) = 1$, hence, $\mathscr{C}(\mathbb{C}, x)$ is not empty. It is easy to see that if $\mathbb{C}$ is crisp, the neighborhood system $\mathscr{C}(\mathbb{C}, x)$ given in Eq. (2.12) is obtained.

The fuzzy minimal and maximal descriptions of x are obtained as follows: in every non-zero membership degree that is reached by x in $\mathbb{C}$, we take the minimal, respectively maximal, sets. This means that for all $\alpha \in \{K(x) \mid K \in \mathbb{C}, K(x) > 0\}$ there exist $K_1 \in \text{md}(\mathbb{C}, x)$ and $K_2 \in \text{MD}(\mathbb{C}, x)$ with $K_1(x) = K_2(x) = \alpha$.

Definition 6.1.3. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $x \in U$, then the *fuzzy minimal description* of x is given by

$$\text{md}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S(x) = K(x), S \subseteq K \Rightarrow S = K)\} \tag{6.2}$$

and the *fuzzy maximal description* of x is given by

$$\text{MD}(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) \mid (\forall S \in \mathscr{C}(\mathbb{C}, x))(S(x) = K(x), S \supseteq K \Rightarrow S = K)\}. \tag{6.3}$$

If $\mathbb{C}$ is crisp, Eqs. (2.13) and (2.14) are obtained. Note that $\mathscr{C}(\mathbb{C}, x)$, $\text{md}(\mathbb{C}, x)$ and $\text{MD}(\mathbb{C}, x)$ are all collections of fuzzy sets and that both $\text{md}(\mathbb{C}, x)$ and $\text{MD}(\mathbb{C}, x)$ are subsets of $\mathscr{C}(\mathbb{C}, x)$.

We illustrate the fuzzy minimal and maximal descriptions in the following example.

Example 6.1.4. Let $U = \{x, y\}$ and let $\mathbb{C} = \{K_1, K_2, K_3, K_4, K_5\}$ be a fuzzy covering on U with $K_1 = 1/x + 0.5/y$, $K_2 = 0.7/x + 1/y$, $K_3 = 0.7/x + 0.5/y$, $K_4 = 0.7/x + 0.2/y$ and $K_5 = 0/x + 0.5/y$. Then $\mathscr{C}(\mathbb{C}, x) = \{K_1, K_2, K_3, K_4\}$, $\text{md}(\mathbb{C}, x) = \{K_1, K_4\}$ and $\text{MD}(\mathbb{C}, x) = \{K_1, K_2\}$. On the other hand, $\mathscr{C}(\mathbb{C}, y) = \mathbb{C}$, $\text{md}(\mathbb{C}, y) = \{K_2, K_4, K_5\}$ and $\text{MD}(\mathbb{C}, y) = \{K_1, K_2, K_4\}$.

Due to the construction of the fuzzy minimal and maximal descriptions, we can extend Proposition 2.2.3 to the fuzzy setting:

Proposition 6.1.5. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space.

(a) If any descending chain in $\mathbb{C}$ is closed under infimum, i.e., for any set $\{K_i \in \mathbb{C} \mid i \in I\}$ with $K_{i+1} \subseteq K_i$ it holds that $\inf_{i \in I} K_i = \bigcap_{i \in I} K_i \in \mathbb{C}$, then for $K \in \mathscr{C}(\mathbb{C}, x)$ there exists $K_1 \in \mathrm{md}(\mathbb{C}, x)$ with $K_1(x) = K(x)$ and $K_1 \subseteq K$.

(b) If any ascending chain in $\mathbb{C}$ is closed under supremum, i.e., for any set $\{K_i \in \mathbb{C} \mid i \in I\}$ with $K_i \subseteq K_{i+1}$ it holds that $\sup_{i \in I} K_i = \bigcup_{i \in I} K_i \in \mathbb{C}$, then for $K \in \mathscr{C}(\mathbb{C}, x)$ there exists $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ with $K_2(x) = K(x)$ and $K \subseteq K_2$.

Proof. (a) Since $K \in \mathscr{C}(\mathbb{C}, x)$, $K(x) > 0$. If $K \notin \mathrm{md}(\mathbb{C}, x)$, then by definition, there exists $K^1 \in \mathbb{C}$ with $K^1(x) = K(x)$ and $K^1 \subsetneq K$. If $K^1 \notin \mathrm{md}(\mathbb{C}, x)$, then by definition, there exists $K^2 \in \mathbb{C}$ with $K^2(x) = K^1(x) = K(x)$ and $K^2 \subsetneq K^1 \subsetneq K$. Continuing, as descending chains in $\mathbb{C}$ are closed under infimum, there exists $K_1 \in \mathbb{C}$ such that $K_1 = \inf_{i \in \mathbb{N}} K^i$ with $K_1(x) = K(x)$ and $K_1(y) \leq K(y)$ for all $y \neq x$ and there is no set smaller in $\mathbb{C}$ than K_1 with these properties, therefore, $K_1 \in \mathrm{md}(\mathbb{C}, x)$.

(b) Similarly, we can find a $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ such that $K_2(x) = K(x)$ and $K \subseteq K_2$. □

Under the hypothesis of Proposition 6.1.5, there always exists a $K_1 \in \mathrm{md}(\mathbb{C}, x)$ and $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ such that $K_1(x) = K_2(x) = 1$. Note that the condition of Proposition 6.1.5 holds whenever the fuzzy covering $\mathbb{C}$ is finite. As this proposition is a motivation for the fuzzy minimal and maximal descriptions, we will often explicitly assume that $\mathbb{C}$ is finite in order to apply this property. Although the condition of a finite $\mathbb{C}$ is stronger than the condition on $\mathbb{C}$ provided in Proposition 6.1.5, we will often assume the former, as it is a more suitable condition for applications. Note that the condition on $\mathbb{C}$ in Proposition 6.1.5 is necessary, as illustrated in the next example:

Example 6.1.6. Let $U = \{x, y\}$ with

$$\mathbb{C} = \{K_n \mid n \in \mathbb{N} \setminus \{0\}\} \cup \{K^* = 0.7/x + 0.1/y\}$$

such that $K_n(x) = 1$ and $K_n(y) = \frac{1}{n}$. As K^* is the only set in $\mathbb{C}$ with $K^*(x) = 0.7$, $K^* \in \mathrm{md}(\mathbb{C}, x)$. However, as for all $n \in \mathbb{N} \setminus \{0\}$ it holds that $K_{n+1} \subseteq K_n$, we obtain that $K_n \notin \mathrm{md}(\mathbb{C}, x)$ for all $n \in \mathbb{N} \setminus \{0\}$. Therefore, there is no set K in $\mathrm{md}(\mathbb{C}, x)$ with $K(x) = 1$.

6.2 Fuzzy neighborhood operators based on a fuzzy covering

In this section, we discuss the notion of fuzzy neighborhood operators in the context of fuzzy rough set theory and we propose definitions for the fuzzy extensions of the four neighborhood operators discussed by Yao and Yao [189].

In the most general setting, a fuzzy neighborhood operator is defined as follows:

Definition 6.2.1. A fuzzy neighborhood operator on U is a mapping

$$\mathrm{N}\colon U \to \mathscr{F}(U),$$

i.e., it associates each object $x \in U$ with a fuzzy set $\mathrm{N}(x)$.

Note that every crisp neighborhood operator is also a fuzzy neighborhood operator, since $\mathscr{P}(U) \subseteq \mathscr{F}(U)$. Therefore, we will use the same notation for a crisp and a fuzzy neighborhood operator. From now on, we assume that N represents a fuzzy neighborhood operator, unless explicitly indicated otherwise.

Fuzzy neighborhood operators on U are in correspondence with fuzzy binary relations on U, just by taking $\mathrm{N}(x)(y) = R(x, y)$ for all $x, y \in U$.

A fuzzy neighborhood operator N on U can satisfy the following properties:

- N is normalized if and only if $\forall x \in U\colon \mathrm{N}(x)$ is normalized, i.e.,

$$\exists y \in U\colon \mathrm{N}(x)(y),$$

- N is reflexive if and only if $\forall x \in U\colon \mathrm{N}(x)(x) = 1$,
- N is symmetric if and only if $\forall x, y \in U\colon \mathrm{N}(x)(y) = \mathrm{N}(y)(x)$,
- given a t-norm $\mathscr{T}$, N is $\mathscr{T}$-transitive if and only if $\forall x, y, z \in U\colon$

$$\mathscr{T}(\mathrm{N}(x)(y), \mathrm{N}(y)(z)) \leq \mathrm{N}(x)(z). \tag{6.4}$$

We will assume that each fuzzy neighborhood operator is reflexive, to fulfil the intuitive idea of a neighborhood. If N is symmetric, then the degree in which an object $y \in U$ belongs to the fuzzy neighborhood of the object $x \in U$ equals the degree in which x belongs to the fuzzy neighborhood of y. If N is $\mathcal{T}_M$-transitive, then the membership degree of the object $z \in U$ belonging to the fuzzy neighborhood of the object $x \in U$ will be at least equal to the minimum of the membership degree of y to $\mathrm{N}(x)$ and the membership degree of z to $\mathrm{N}(y)$, for each element $y \in U$.

Given the definitions of the fuzzy neighborhood system and the fuzzy minimal and maximal description of an object $x \in U$ for a given fuzzy covering $\mathbb{C}$, we can now introduce fuzzy extensions of the four crisp neighborhood operators $N_1^{\mathbb{C}}, N_2^{\mathbb{C}}, N_3^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ defined in [189].

Fuzzy neighborhood operator $N_1^{\mathbb{C}}$

To introduce a fuzzy extension for the crisp neighborhood operator $N_1^{\mathbb{C}}$, we can rewrite the condition $y \in \bigcap \mathscr{C}(\mathbb{C}, x)$ for a crisp covering $\mathbb{C}$ as

$$\forall K \in \mathbb{C} \colon x \in K \Rightarrow y \in K.$$

A natural extension of this definition follows from replacing $\forall$ by the infimum operator, $x \in K$ by the membership degree $K(x)$ and $\Rightarrow$ by an implicator $\mathscr{I}$.

Definition 6.2.2. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{I}$ an implicator, then

$$N_1^{\mathbb{C}} \colon U \to \mathscr{F}(U) \colon x \mapsto N_1^{\mathbb{C}}(x) \tag{6.5}$$

is a fuzzy neighborhood operator, such that for $x \in U$ the fuzzy neighborhood $N_1^{\mathbb{C}}(x)$ is defined by

$$N_1^{\mathbb{C}}(x) \colon U \to [0,1] \colon y \mapsto \inf_{K \in \mathbb{C}} \mathscr{I}(K(x), K(y)). \tag{6.6}$$

If the covering $\mathbb{C}$ is a crisp covering, then the fuzzy neighborhood of x described in Eq. (6.6) coincides with the crisp neighborhood $N_1^{\mathbb{C}}(x)$. Indeed, for $x, y \in U$, the membership degree $N_1^{\mathbb{C}}(x)(y)$ is either 0 or 1 if the covering $\mathbb{C}$ is crisp. Moreover, it holds that

$$N_1^{\mathbb{C}}(x)(y) = 1 \quad \Leftrightarrow \quad \inf_{K \in \mathbb{C}} \mathscr{I}(K(x), K(y)) = 1$$

$$\begin{aligned}
&\Leftrightarrow \quad \forall K \in \mathbb{C}\colon \mathcal{I}(K(x),K(y)) = 1\\
&\Leftrightarrow \quad \forall K \in \mathbb{C}\colon K(x) = 1 \Rightarrow K(y) = 1\\
&\Leftrightarrow \quad \forall K \in \mathbb{C}\colon x \in K \Rightarrow y \in K\\
&\Leftrightarrow \quad y \in \bigcap \mathscr{C}(\mathbb{C}, x).
\end{aligned}$$

To construct the fuzzy neighborhoood operator $N_1^{\mathbb{C}}$, we have used the characterization $y \in \bigcap \mathscr{C}(\mathbb{C}, x)$ of the crisp neighborhood $N_1^{\mathbb{C}}(x)$. Next, we prove that the characterization $y \in \bigcap \mathrm{md}(\mathbb{C}, x)$ yields the same fuzzy neighborhood operator for a finite fuzzy covering $\mathbb{C}$.

Proposition 6.2.3. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $\mathcal{I}$ an implicator, then for all $x, y \in U$ it holds that

$$\inf_{K\in\mathbb{C}} \mathcal{I}(K(x),K(y)) = \inf_{K\in\mathscr{C}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)) = \inf_{K\in\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)).$$

Proof. First note that if $K(x) = 0$, then $\mathcal{I}(K(x),K(y)) = 1$, hence

$$\inf_{K\in\mathbb{C}} \mathcal{I}(K(x),K(y)) = \inf_{K\in\mathscr{C}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)).$$

Since $\mathrm{md}(\mathbb{C}, x) \subseteq \mathscr{C}(\mathbb{C}, x)$, we have that

$$\inf_{K\in\mathbb{C}} \mathcal{I}(K(x),K(y)) = \\ \min\left(\inf_{K\in\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)), \inf_{K\in\mathscr{C}(\mathbb{C},x)\setminus\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y))\right).$$

If $K \in \mathscr{C}(\mathbb{C}, x) \setminus \mathrm{md}(\mathbb{C}, x)$, then there exists a $K' \in \mathrm{md}(\mathbb{C}, x)$ such that $K' \subseteq K$ and $K'(x) = K(x)$. Therefore, for all $y \in U$,

$$\mathcal{I}(K(x),K(y)) = \mathcal{I}(K'(x),K(y)) \geq \mathcal{I}(K'(x),K'(y)).$$

Hence, we can conclude that

$$\inf_{K\in\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)) \leq \inf_{K\in\mathscr{C}(\mathbb{C},x)\setminus\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y))$$

and thus,

$$\inf_{K\in\mathbb{C}} \mathcal{I}(K(x),K(y)) = \inf_{K\in\mathrm{md}(\mathbb{C},x)} \mathcal{I}(K(x),K(y)).$$

□

Note that we assume the fuzzy covering $\mathbb{C}$ to be finite, in order to be able to apply Proposition 6.1.5.

Fuzzy neighborhood operator $N_2^{\mathbb{C}}$

For the fuzzy extension of the crisp neighborhood operator $N_2^{\mathbb{C}}$, note that the condition $y \in \bigcup \mathrm{md}(\mathbb{C}, x)$ can be rewritten as

$$\exists K \in \mathrm{md}(\mathbb{C}, x)\colon x \in K \wedge y \in K,$$

when $\mathbb{C}$ is a crisp covering. A natural extension of this definition follows from replacing $\exists$ by the supremum operator, $x \in K$ by the membership degree $K(x)$ and $\wedge$ by a t-norm $\mathscr{T}$.

Definition 6.2.4. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{T}$ a t-norm, then

$$N_2^{\mathbb{C}}\colon U \to \mathscr{F}(U)\colon x \mapsto N_2^{\mathbb{C}}(x) \tag{6.7}$$

is a fuzzy neighborhood operator, such that for $x \in U$ the fuzzy neighborhood $N_2^{\mathbb{C}}(x)$ is defined by

$$N_2^{\mathbb{C}}(x)\colon U \to [0,1]\colon y \mapsto \sup_{K \in \mathrm{md}(\mathbb{C}, x)} \mathscr{T}(K(x), K(y)). \tag{6.8}$$

Note that if the covering $\mathbb{C}$ is a crisp covering, the fuzzy minimal description of x coincides with the crisp minimal description of x. Moreover, in that case, the fuzzy neighborhood defined in Eq. (6.8) coincides with the crisp neighborhood $N_2^{\mathbb{C}}(x)$.

Fuzzy neighborhood operator $N_3^{\mathbb{C}}$

For a crisp covering $\mathbb{C}$, the condition $y \in \bigcap \mathrm{MD}(\mathbb{C}, x)$ can be rewritten as

$$\forall K \in \mathrm{MD}(\mathbb{C}, x)\colon x \in K \Rightarrow y \in K.$$

As with the operator $N_1^{\mathbb{C}}$, a natural extension of this definition follows from replacing $\forall$ by the infimum operator, $x \in K$ by the membership degree $K(x)$ and $\Rightarrow$ by an implicator $\mathscr{I}$.

Definition 6.2.5. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{I}$ an implicator, then

$$N_3^{\mathbb{C}}\colon U \to \mathscr{F}(U)\colon x \mapsto N_3^{\mathbb{C}}(x) \tag{6.9}$$

is a fuzzy neighborhood operator, such that for $x \in U$ the fuzzy neighborhood $N_3^{\mathbb{C}}(x)$ is defined by

$$N_3^{\mathbb{C}}(x)\colon U \to [0,1]\colon y \mapsto \inf_{K \in \mathrm{MD}(\mathbb{C},x)} \mathscr{I}(K(x), K(y)). \tag{6.10}$$

Given a crisp covering $\mathbb{C}$, the fuzzy maximal description of x coincides with the crisp maximal description of x and the fuzzy neighborhood of x described in Eq. (6.10) coincides with the crisp neighborhood $N_3^{\mathbb{C}}(x)$.

Fuzzy neighborhoood operator $N_4^{\mathbb{C}}$

We can rewrite the characterization $y \in \bigcup \mathscr{C}(\mathbb{C}, x)$ for a crisp covering $\mathbb{C}$ as

$$\exists K \in \mathbb{C}\colon x \in K \wedge y \in K.$$

As with the operator $N_2^{\mathbb{C}}$, a natural extension of this definition follows from replacing $\exists$ by the supremum operator, $x \in K$ by the membership degree $K(x)$ and $\wedge$ by a t-norm $\mathscr{T}$.

Definition 6.2.6. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{T}$ a t-norm, then

$$N_4^{\mathbb{C}}\colon U \to \mathscr{F}(U)\colon x \mapsto N_4^{\mathbb{C}}(x) \tag{6.11}$$

is a fuzzy neighborhood operator, such that for $x \in U$ the fuzzy neighborhood $N_4^{\mathbb{C}}(x)$ is defined by

$$N_4^{\mathbb{C}}(x)\colon U \to [0,1]\colon y \mapsto \sup_{K \in \mathbb{C}} \mathscr{T}(K(x), K(y)). \tag{6.12}$$

It is easy to see that the fuzzy neighborhood of x defined in Eq. (6.12) is a fuzzy extension of the crisp neighborhood $N_4^{\mathbb{C}}(x)$.

We can prove the analogy of the crisp equality

$$\bigcup \mathscr{C}(\mathbb{C}, x) = \bigcup \mathrm{MD}(\mathbb{C}, x)$$

for a finite fuzzy covering $\mathbb{C}$.

Proposition 6.2.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $\mathscr{T}$ a t-norm, then for all $x,y \in U$ it holds that

$$\sup_{K\in\mathbb{C}} \mathscr{T}(K(x),K(y)) = \sup_{K\in\mathscr{C}(\mathbb{C},x)} \mathscr{T}(K(x),K(y)) = \sup_{K\in\mathrm{MD}(\mathbb{C},x)} \mathscr{T}(K(x),K(y)).$$

Proof. Analogously as in the proof of Proposition 6.2.3, we can prove that

$$\sup_{K\in\mathbb{C}} \mathscr{T}(K(x),K(y)) = \sup_{K\in\mathscr{C}(\mathbb{C},x)} \mathscr{T}(K(x),K(y))$$

and

$$\sup_{K\in\mathrm{MD}(\mathbb{C},x)} \mathscr{T}(K(x),K(y)) \geq \sup_{K\in\mathscr{C}(\mathbb{C},x)\setminus\mathrm{MD}(\mathbb{C},x)} \mathscr{T}(K(x),K(y)).$$

Hence, the supremum will be reached in $\mathrm{MD}(\mathbb{C},x)$. □

To end this section, note that the four fuzzy neighborhood operators are reflexive under certain conditions:

Proposition 6.2.8. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{T}$ a t-norm and $\mathscr{I}$ a WCP-implicator, then the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ defined with $\mathscr{I}$ and the fuzzy neighborhood operator $N_4^{\mathbb{C}}$ defined with $\mathscr{T}$ are reflexive fuzzy neighborhood operators. The fuzzy neighborhood operator $N_2^{\mathbb{C}}$ defined with $\mathscr{T}$ is reflexive if $\mathbb{C}$ is finite.

Proof. As $\mathscr{I}(a,a) = 1$ for all $a \in [0,1]$, the operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are reflexive. Moreover, let $x \in U$, then there exists $K \in \mathbb{C}$ such that $K(x) = 1$. Hence, $N_4^{\mathbb{C}}(x)(x) \geq \mathscr{T}(K(x),K(x)) = 1$.

Now assume $\mathbb{C}$ is finite, then there exists $K_1 \in \mathrm{md}(\mathbb{C},x)$ with $K_1(x) = K(x) = 1$ and $K_1 \subseteq K$. Hence, $N_2^{\mathbb{C}}(x)(x) \geq \mathscr{T}(K_1(x),K_1(x)) = 1$, and thus, $N_2^{\mathbb{C}}$ is a reflexive fuzzy neighborhood operator. □

6.3 Fuzzy coverings derived from a fuzzy covering

For a crisp covering $\mathbb{C}$, Yao and Yao [189] introduced six derived coverings of $\mathbb{C}$: $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap$ and $\mathbb{C}_\cup$. In this section, we extend these derived coverings to the fuzzy setting.

Definition 6.3.1. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, consider the t-norm $\mathscr{T}$ to construct the fuzzy neighborhood operator $N_4^{\mathbb{C}}$ and consider the implicator $\mathscr{I}$ to construct the fuzzy neighborhood operator $N_1^{\mathbb{C}}$, then define the following collections of fuzzy sets:

$$\mathbb{C}_1 = \bigcup\{\mathrm{md}(\mathbb{C},x) \mid x \in U\}, \tag{6.13}$$
$$\mathbb{C}_2 = \bigcup\{\mathrm{MD}(\mathbb{C},x) \mid x \in U\}, \tag{6.14}$$
$$\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x) \mid x \in U\}, \tag{6.15}$$
$$\mathbb{C}_4 = \{N_4^{\mathbb{C}}(x) \mid x \in U\}, \tag{6.16}$$
$$\mathbb{C}_\cap = \mathbb{C} \setminus \{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})(K = \bigcap \mathbb{C}')\}, \tag{6.17}$$
$$\mathbb{C}_\cup = \mathbb{C} \setminus \{K \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\})(K = \bigcup \mathbb{C}')\}. \tag{6.18}$$

We illustrate these definitions in the following example.

Example 6.3.2. Let $U = \{x,y,z\}$ and $\mathbb{C} = \{K_1,K_2,K_3,K_4,K_5,K_6,K_7,K_8\}$ with $K_1 = 0.9/x+0.9/y+0.7/z$, $K_2 = 0.9/x+1/y+0.9/z$, $K_3 = 0.9/x+0.1/y+0.8/z$, $K_4 = 1/x + 0.9/y + 1/z$, $K_5 = 1/x + 0/y + 0.6/z$, $K_6 = 1/x + 1/y + 0.7/z$, $K_7 = 1/x+0/y+0.5/z$ and $K_8 = 1/x+0.9/y+0.7/z$, then

$$\mathbb{C}_1 = \mathbb{C}_\cup = \{K_1,K_2,K_3,K_4,K_5,K_6,K_7\}$$

and

$$\mathbb{C}_2 = \mathbb{C}_\cap = \{K_2,K_3,K_4,K_5,K_6,K_7\}.$$

Moreover, if $\mathscr{T}$ is the minimum operator and $\mathscr{I}$ its R-implicator, then

$$\mathbb{C}_3 = \{1/x+0/y+0.5/z, 0.9/x+1/y+0.7/z, 1/x+0/y+1/z\}$$

and

$$\mathbb{C}_4 = \{1/x+1/y+1/z, 1/x+1/y+0.9/z, 1/x+0.9/y+1/z\}.$$

Next, we will prove that the collections defined above are all fuzzy coverings if the original fuzzy covering $\mathbb{C}$ is finite. First, we show that the collections $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cup$ are finite fuzzy subcoverings of $\mathbb{C}$.

Proposition 6.3.3. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, then $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cup$ are all finite fuzzy subcoverings of $\mathbb{C}$.

Proof. It is easy to see that all three collections are subsets of the fuzzy covering $\mathbb{C}$, and that they are finite collections of non-empty fuzzy sets of U. We need to study for $\mathbb{C}_j \in \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_\cup\}$ whether the condition

$$\forall x \in U, \exists K_j \in \mathbb{C}_j : K_j(x) = 1$$

holds.

Take $x \in U$, then there exists $K \in \mathbb{C}$ such that $K(x) = 1$. Hence, there exist $K_1 \in \mathrm{md}(\mathbb{C}, x)$ and $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ such that $K_1(x) = K(x) = K_2(x) = 1$ and $K_1 \subseteq K \subseteq K_2$. Since $K_1 \in \mathbb{C}_1$ and $K_2 \in \mathbb{C}_2$, $\mathbb{C}_1$ and $\mathbb{C}_2$ are fuzzy coverings.

As for $\mathbb{C}_\cup$, assume $K \notin \mathbb{C}_\cup$, then there exists a collection $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ such that $K = \bigcup \mathbb{C}'$. Since $\mathbb{C}$ is finite, there exists a $K' \in \mathbb{C}'$ such that $K'(x) = 1$. Since we can choose the collection $\mathbb{C}'$ in $\mathbb{C}_\cup$, there exists $K' \in \mathbb{C}_\cup$ such that $K'(x) = 1$. Hence, $\mathbb{C}_\cup$ is a fuzzy covering. □

The condition of finiteness for $\mathbb{C}$ is necessary, as for the coverings $\mathbb{C}_1$ and $\mathbb{C}_2$ Proposition 6.1.5 is used. The necessity of the condition for $\mathbb{C}_\cup$ is illustrated in the next example:

Example 6.3.4. Let $U = \{x\}$ and $\mathbb{C} = \{K_n \mid n \in \mathbb{N} \setminus \{0\}\} \cup \{K^*\}$ with $K_n(x) = 1 - \frac{1}{n}$ and $K^*(x) = 1$. It holds that $\sup\{K_n(x) \mid \mathbb{N} \setminus \{0\}\} = K^*(x)$, and thus, $K^* \notin \mathbb{C}_\cup$. Therefore, $\mathbb{C}_\cup$ is not a fuzzy covering.

Next, we show that $\mathbb{C}_\cap$ is a fuzzy covering for an infinite covering $\mathbb{C}$.

Proposition 6.3.5. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, then $\mathbb{C}_\cap$ is a fuzzy subcovering of $\mathbb{C}$.

Proof. By definition, $\mathbb{C}_\cap$ is a subset of $\mathbb{C}$ and therefore, it does not contain the empty set. Moreover, let $x \in U$ and $K \in \mathbb{C}$ with $K(x) = 1$. Assume that $K \notin \mathbb{C}_\cap$, then there exists a collection $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ such that $K = \bigcap \mathbb{C}'$. Since $K(x) = 1$, it holds for all $K' \in \mathbb{C}'$ that $K'(x) = 1$. Since we can choose the collection $\mathbb{C}'$ in $\mathbb{C}_\cap$, there exists $K' \in \mathbb{C}_\cap$ such that $K'(x) = 1$. Hence, $\mathbb{C}_\cap$ is a fuzzy covering. □

Furthermore, we prove that $\mathbb{C}_3$ and $\mathbb{C}_4$ are fuzzy coverings.

Proposition 6.3.6. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{T}$ a t-norm to construct $\mathbb{C}_4$ and $\mathscr{I}$ a WCP-implicator to construct $\mathbb{C}_3$, then $\mathbb{C}_3$ and $\mathbb{C}_4$ are fuzzy coverings.

Proof. This follows immediately from the fact that $N_1^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ are reflexive (see Proposition 6.2.8). □

As opposed to the fuzzy coverings $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_\cap$ and $\mathbb{C}_\cup$ which are subcoverings of $\mathbb{C}$, it is possible that the fuzzy coverings $\mathbb{C}_3$ and $\mathbb{C}_4$ have no overlap with the original covering $\mathbb{C}$. Moreover, note that the cardinality of $\mathbb{C}_3$ and $\mathbb{C}_4$ is at most equal to the cardinality of U, while the cardinality of the other four derived coverings will be at most equal to the cardinality of $\mathbb{C}$.

In the following, we prove that $\mathbb{C}_2 \subseteq \mathbb{C}_\cap$ and $\mathbb{C}_\cup = \mathbb{C}_1$ if $\mathbb{C}$ is finite.

Proposition 6.3.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, then $\mathbb{C}_2$ is a fuzzy subcovering of $\mathbb{C}_\cap$.

Proof. Let $K \in \mathbb{C}_2$, then there exists $x \in U$ such that $K \in \mathrm{MD}(\mathbb{C}, x)$. If $K \notin \mathbb{C}_\cap$, then there exists a collection $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ such that $K = \bigcap \mathbb{C}'$. We can choose the collection $\mathbb{C}'$ in $\mathbb{C}_\cap$. Since $\mathbb{C}$ is finite, take $K' \in \mathbb{C}'$ such that $K'(x) = K(x) > 0$. Because $K \subseteq K'$, $K(x) = K'(x)$ and $K \in \mathrm{MD}(\mathbb{C}, x)$, we have that $K = K'$. Hence, $K \in \mathbb{C}_\cap$. □

Note that $\mathbb{C}_2$ is not necessarily a subset of $\mathbb{C}_\cap$ if $\mathbb{C}$ is infinite:

Example 6.3.8. Let $U = \{x\}$ and the covering $\mathbb{C} = \{K_n \mid n \in \mathbb{N} \setminus \{0,1\}\} \cup \{K^*\}$ is defined by $K_n(x) = \frac{1}{2} + \frac{1}{n}$ and $K^*(x) = \frac{1}{2}$, then

$$\inf\{K_n(x) \mid n \in \mathbb{N} \setminus \{0,1\}\} = K^*(x),$$

thus $K^* \notin \mathbb{C}_\cap$. Since the membership degree of x in every fuzzy set of $\mathbb{C}$ is different, we have that $\mathbb{C}_2 = \mathbb{C}$ and therefore, $\mathbb{C}_2$ is not a subset of $\mathbb{C}_\cap$.

Proposition 6.3.9. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, then $\mathbb{C}_\cup = \mathbb{C}_1$.

Proof. First, let $K \in \mathbb{C}_1$, then there exists $x \in U$ such that $K \in \text{md}(\mathbb{C}, x)$. If $K \notin \mathbb{C}_\cup$, then there exists a collection $\mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}$ such that $K = \bigcup \mathbb{C}'$. We can choose the collection $\mathbb{C}'$ in $\mathbb{C}_\cup$. Since $\mathbb{C}$ is finite, take $K' \in \mathbb{C}'$ such that $K'(x) = K(x) > 0$. Because $K' \subseteq K$, $K'(x) = K(x)$ and $K \in \text{md}(\mathbb{C}, x)$, we have that $K = K'$. Hence, $K \in \mathbb{C}_\cup$.

Second, let $K \in \mathbb{C}_\cup$ and assume that $K \notin \mathbb{C}_1$, then for all $x \in U$, $K \notin \text{md}(\mathbb{C}, x)$. Since K is not empty, there exists $x \in U$ such that $K(x) > 0$. Hence, there exists $K_x \in \text{md}(\mathbb{C}, x)$ with $K_x(x) = K(x)$ and $K_x \subsetneq K$. Therefore,

$$\bigcup \{K_x \mid x \in U, K_x \in \text{md}(\mathbb{C}, x) \colon K_x(x) = K(x) > 0, K_x \subseteq K\} \subseteq K.$$

On the other hand, for each $z \in U$ we have that

$$\sup_{x \in U \colon K(x) > 0} K_x(z) \geq K(z),$$

because if $K(z) > 0$, then $\sup_{x \in U \colon K(x) > 0} K_x(z) \geq K_z(z) = K(z)$ and if $K(z) = 0$, it holds trivially. Hence, we conclude that

$$K = \bigcup \{K_x \mid x \in U, K_x \in \text{md}(\mathbb{C}, x) \colon K_x(x) = K(x) > 0, K_x \subseteq K\},$$

where $\{K_x \mid x \in U, K(x) > 0\} \subseteq \mathbb{C} \setminus \{K\}$, which means that $K \notin \mathbb{C}_\cup$. This is a contradiction, thus, $K \in \mathbb{C}_1$. □

Note that the finiteness condition is necessary, as otherwise $\mathbb{C}_1$ and $\mathbb{C}_\cup$ are not fuzzy coverings.

We can conclude that a finite fuzzy covering $\mathbb{C}$ yields five derived fuzzy coverings $\mathbb{C}_1 = \mathbb{C}_\cup$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_4$ and $\mathbb{C}_\cap$. These six fuzzy coverings (one original and five derived ones) together with the four fuzzy neighborhood operators result in twenty-four combinations of fuzzy neighborhood operators based on a finite fuzzy covering.

6.4 Equalities between fuzzy neighborhood operators

In this section, we discuss equalities between fuzzy neighborhood operators based on a fuzzy covering $\mathbb{C}$. We will assume that this fuzzy covering $\mathbb{C}$ is finite, therefore, the fuzzy covering $\mathbb{C}_\cup$ is disregarded since it is equal to the fuzzy covering $\mathbb{C}_1$. First note that when two neighborhood operators are different in the crisp case, they are also different in the fuzzy setting. Hence, if we consider the fuzzy neighborhood operators $N_i^{\mathbb{C}_j}$ for $i \in \{1,2,3,4\}$ and $\mathbb{C}_j \in \{\mathbb{C},\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3,\mathbb{C}_4,\mathbb{C}_\cap\}$, we only need to study whether the equalities of the neighborhood operators of groups a, c, f and j of Table 4.1 are maintained.

We start with the following observations.

Proposition 6.4.1. Let $(U,\mathbb{C})$ be a fuzzy covering approximaton space with $\mathbb{C}$ finite, then for all $x \in U$ it holds that

(a) $\mathrm{md}(\mathbb{C}_1, x) = \mathrm{md}(\mathbb{C}, x)$,

(b) $\mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}, x)$,

(c) $\mathrm{MD}(\mathbb{C}_\cap, x) = \mathrm{MD}(\mathbb{C}, x)$.

Proof. (a) Take $x \in U$. If $K \in \mathrm{md}(\mathbb{C}, x)$, then $K \in \mathbb{C}_1$ and $K(x) > 0$. Let $K' \in \mathbb{C}_1$ with $K'(x) = K(x) > 0$ and $K' \subseteq K$. Since $K' \in \mathbb{C}$ and $K \in \mathrm{md}(\mathbb{C}, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{md}(\mathbb{C}_1, x)$.

On the other hand, if $K \in \mathrm{md}(\mathbb{C}_1, x)$, then $K \in \mathbb{C}_1 \subseteq \mathbb{C}$ and $K(x) > 0$. Hence, there exists $K' \in \mathrm{md}(\mathbb{C}, x)$ with $K'(x) = K(x)$ and $K' \subseteq K$. Since $K' \in \mathbb{C}_1$ and $K \in \mathrm{md}(\mathbb{C}_1, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{md}(\mathbb{C}, x)$.

(b) Take $x \in U$. If $K \in \mathrm{MD}(\mathbb{C}, x)$, then $K \in \mathbb{C}_2$ and $K(x) > 0$. Let $K' \in \mathbb{C}_2$ with $K'(x) = K(x) > 0$ and $K \subseteq K'$. Since $K' \in \mathbb{C}$ and $K \in \mathrm{MD}(\mathbb{C}, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}_2, x)$.

On the other hand, if $K \in \mathrm{MD}(\mathbb{C}_2, x)$, then $K \in \mathbb{C}_2 \subseteq \mathbb{C}$ and $K(x) > 0$. Hence, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K'(x) = K(x)$ and $K \subseteq K'$. Since $K' \in \mathbb{C}_2$ and $K \in \mathrm{MD}(\mathbb{C}_2, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}, x)$.

(c) Let $\mathbb{C}$ be finite, then $\mathbb{C}_2$ is a fuzzy subcovering of $\mathbb{C}_\cap$, and $x \in U$. On the one hand, if $K \in \mathrm{MD}(\mathbb{C}, x)$, then $K \in \mathbb{C}_2 \subseteq \mathbb{C}_\cap$ and $K(x) > 0$. Let $K' \in \mathbb{C}_\cap$ with $K'(x) = K(x) > 0$ and $K \subseteq K'$. Since $K' \in \mathbb{C}$ and $K \in \mathrm{MD}(\mathbb{C}, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$.

On the other hand, if $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$, then $K \in \mathbb{C}_\cap \subseteq \mathbb{C}$ and $K(x) > 0$. Hence, there exists $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K'(x) = K(x)$ and $K \subseteq K'$. Since $K' \in \mathbb{C}_2 \subseteq \mathbb{C}_\cap$ and $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$, it holds that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}, x)$. □

Note that in the fuzzy setting the equality $\mathrm{md}(\mathbb{C}_2, x) = \mathscr{C}(\mathbb{C}, x)$ and the equality $\mathrm{MD}(\mathbb{C}_2, x) = \mathscr{C}(\mathbb{C}, x)$ no longer hold as illustrated in the next example.

Example 6.4.2. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.5/y$ and $K_2 = 1/x + 1/y$, then $\mathbb{C}_2 = \mathbb{C}$. We have that $\mathscr{C}(\mathbb{C}_2, x) = \mathscr{C}(\mathbb{C}, x) = \{K_1, K_2\}$, $\mathrm{md}(\mathbb{C}_2, x) = \{K_1\}$ and $\mathrm{MD}(\mathbb{C}_2, x) = \{K_2\}$.

The first group we discuss contains the fuzzy neighborhood operators $N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$, $N_1^{\mathbb{C}_\cap}$ and $N_2^{\mathbb{C}_3}$. We show that the first four fuzzy neighborhood operators are still equal in the fuzzy setting, but the last fuzzy neighborhood operator is different.

Proposition 6.4.3. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $\mathscr{I}$ an implicator used to define the covering $\mathbb{C}_3$ and the fuzzy neighborhood operators $N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$ and $N_1^{\mathbb{C}_\cap}$, then

(a) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1}$,

(b) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_3}$ if $\mathscr{I}$ is the R-implicator of a left-continuous t-norm,

(c) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_\cap}$.

Proof. (a) This follows immediately from Propositions 6.2.3 and 6.4.1.

(b) Assume that $\mathscr{I}$ is an R-implicator of a left-continuous t-norm $\mathscr{T}$. Since $N_1^{\mathbb{C}}(x) \in \mathbb{C}_3$, we have for $y \in U$ that

$$\begin{aligned} N_1^{\mathbb{C}_3}(x)(y) &\leq \mathscr{I}(N_1^{\mathbb{C}}(x)(x), N_1^{\mathbb{C}}(x)(y)) \\ &= \mathscr{I}(1, N_1^{\mathbb{C}}(x)(y)) \\ &= N_1^{\mathbb{C}}(x)(y). \end{aligned}$$

On the other hand, by Proposition 6.5.13, $N_1^{\mathbb{C}}$ is $\mathscr{T}$-transitive. Therefore, for all $z \in U$, we have

$$\mathscr{T}(N_1^{\mathbb{C}}(z)(x), N_1^{\mathbb{C}}(x)(y)) \leq N_1^{\mathbb{C}}(z)(y)$$

$$\begin{aligned} &\Rightarrow \quad \mathcal{T}(N_1^{\mathbb{C}}(x)(y), N_1^{\mathbb{C}}(z)(x)) \leq N_1^{\mathbb{C}}(z)(y) \\ &\Rightarrow \quad N_1^{\mathbb{C}}(x)(y) \leq \mathcal{I}(N_1^{\mathbb{C}}(z)(x), N_1^{\mathbb{C}}(z)(y)), \end{aligned}$$

by the residuation priniciple. Hence,

$$\begin{aligned} N_1^{\mathbb{C}}(x)(y) &\leq \inf_{z\in U} \mathcal{I}(N_1^{\mathbb{C}}(z)(x), N_1^{\mathbb{C}}(z)(y)) \\ &= \inf_{K\in\mathbb{C}_3} \mathcal{I}(K(x), K(y)) \\ &= N_1^{\mathbb{C}_3}(x)(y). \end{aligned}$$

(c) Since $\mathbb{C}_\cap \subseteq \mathbb{C}$, $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_\cap}(x)$ for all $x \in U$.

On the other hand, take $y \in U$. Since $\mathbb{C}$ is finite, let $K \in \mathbb{C}$ such that $N_1^{\mathbb{C}}(x)(y) = \mathcal{I}(K(x), K(y))$. If $K \in \mathbb{C}_\cap$, then

$$N_1^{\mathbb{C}_\cap}(x)(y) \leq \mathcal{I}(K(x), K(y)) = N_1^{\mathbb{C}}(x)(y).$$

If $K \notin \mathbb{C}_\cap$, we can find a collection $\mathbb{C}' \subseteq \mathbb{C}_\cap$ such that $K = \bigcap \mathbb{C}'$. Since $\mathbb{C}$ is finite, there exists $K' \in \mathbb{C}'$ with $K'(y) = K(y)$ and $K \subseteq K'$. Therefore,

$$\mathcal{I}(K(x), K(y)) \geq \mathcal{I}(K'(x), K'(y)).$$

Moreover, we also have that

$$\mathcal{I}(K(x), K(y)) \leq \mathcal{I}(K'(x), K'(y))$$

since the infimum of $N_1^{\mathbb{C}}(x)(y)$ is reached in K and $K' \in \mathbb{C}$. Therefore,

$$N_1^{\mathbb{C}_\cap}(x)(y) \leq \mathcal{I}(K'(x), K'(y)) = \mathcal{I}(K(x), K(y)) = N_1^{\mathbb{C}}(x)(y).$$

In both cases we can conclude that $N_1^{\mathbb{C}_\cap}(x) \subseteq N_1^{\mathbb{C}}(x)$.

□

In the following example, we illustrate that the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_2^{\mathbb{C}_3}$ are, in general, no longer equal.

Example 6.4.4. Let $U = \{x, y, z\}$ and let $\mathbb{C} = \{K_1, K_2, K_3\}$ be a fuzzy covering on U with $K_1 = 1/x + 0.8/y + 0.6/z$, $K_2 = 0.2/x + 1/y + 0.6/z$, $K_3 = 0.2/x + 0.8/y + 1/z$. Let $\mathcal{T}$ be the minimum operator and $\mathcal{I}$ its R-implicator, then $N_1^{\mathbb{C}}(y)(x) = 0.2$. On the other hand, $\mathbb{C}_3 = \mathbb{C}$ and $N_2^{\mathbb{C}_3}(y)(x) = 0.8$.

Next, we consider the group with fuzzy neighborhood operators $N_2^{\mathbb{C}}$ and $N_2^{\mathbb{C}_1}$. These operators are still equal in the fuzzy setting.

Proposition 6.4.5. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $N_2^{\mathbb{C}}$ and $N_2^{\mathbb{C}_1}$ based on the t-norm $\mathscr{T}$, then $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$.

Proof. This follows immediately from Proposition 6.4.1. □

The third group we discuss, is the group containing the fuzzy neighborhood operators $N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$ and $N_1^{\mathbb{C}_2}$. The first three fuzzy neighborhood operators are still equal, but the fourth fuzzy neighborhood operator is different in the fuzzy setting.

Proposition 6.4.6. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $N_3^{\mathbb{C}}$, $N_3^{\mathbb{C}_2}$ and $N_3^{\mathbb{C}_\cap}$ based on the implicator $\mathscr{I}$, then

(a) $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_2}$,

(b) $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_\cap}$.

Proof. This follows immediately from Proposition 6.4.1. □

The fuzzy neighborhood operators $N_3^{\mathbb{C}}$ and $N_1^{\mathbb{C}_2}$ are no longer equal.

Example 6.4.7. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.5/y$ and $K_2 = 1/x + 1/y$, then $\mathbb{C}_2 = \mathbb{C}$. Let $\mathscr{I}$ be a border implicator. We have that $N_3^{\mathbb{C}}(x)(y) = \mathscr{I}(K_2(x), K_2(y)) = 1$ and $N_1^{\mathbb{C}_2}(x)(y) = \mathscr{I}(K_1(x), K_1(y)) = 0.5$.

The final group we discuss consists of the fuzzy neighborhood operators $N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$ and $N_2^{\mathbb{C}_2}$. The first three fuzzy neighborhood operators are equal, but the fourth fuzzy neighborhood operator is different in the fuzzy setting.

Proposition 6.4.8. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $N_4^{\mathbb{C}}$, $N_4^{\mathbb{C}_2}$ and $N_4^{\mathbb{C}_\cap}$ based on the t-norm $\mathscr{T}$, then

(a) $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_2}$,

(b) $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_\cap}$.

Proof. This follows immediately from Propositions 6.2.7 and 6.4.1. □

The fuzzy neighborhood operators $N_4^{\mathbb{C}}$ and $N_2^{\mathbb{C}_2}$ are no longer equal.

Example 6.4.9. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.5/y$ and $K_2 = 1/x + 1/y$, then $\mathbb{C}_2 = \mathbb{C}$. Let $\mathcal{T}$ be a t-norm. We have that

$$N_4^{\mathbb{C}}(x)(y) = \mathcal{T}(K_2(x), K_2(y)) = 1$$

and

$$N_2^{\mathbb{C}_2}(x)(y) = \mathcal{T}(K_1(x), K_1(y)) = 0.5.$$

We conclude that given a finite fuzzy covering $\mathbb{C}$, a left-continuous t-norm to construct $\mathbb{C}_4$ and the fuzzy neighborhood operators $N_2^{\mathbb{C}_j}$ and $N_4^{\mathbb{C}_j}$ and its R-implicator to construct $\mathbb{C}_3$ and the fuzzy neighborhood operators $N_1^{\mathbb{C}_j}$ and $N_3^{\mathbb{C}_j}$, we have 16 different groups of fuzzy neighborhood operators, listed in Table 6.1.

Table 6.1: Fuzzy neighborhood operators $N_i^{\mathbb{C}_j}$ for $(U, \mathbb{C})$ with $\mathbb{C}$ finite, $\mathcal{T}$ left-continuous and $\mathcal{I}$ its R-implicator

Group	Operators	Group	Operators
a_1.	$N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}$	g.	$N_4^{\mathbb{C}_3}$
a_2.	$N_2^{\mathbb{C}_3}$	h.	$N_4^{\mathbb{C}_1}$
b.	$N_3^{\mathbb{C}_3}$	i.	$N_1^{\mathbb{C}_4}$
c.	$N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$	j_1.	$N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
d.	$N_3^{\mathbb{C}_1}$	j_2.	$N_2^{\mathbb{C}_2}$
e.	$N_2^{\mathbb{C}_\cap}$	k.	$N_2^{\mathbb{C}_4}$
f_1.	$N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$	l.	$N_3^{\mathbb{C}_4}$
f_2.	$N_1^{\mathbb{C}_2}$	m.	$N_4^{\mathbb{C}_4}$

6.5 Partial order relations between fuzzy neighborhood operators

In this section, we describe the partial order relations between the different groups of fuzzy neighborhood operators of Table 6.1. We assume $\mathbb{C}$ to be a finite fuzzy covering, $\mathcal{T}$ a left-continuous t-norm which is used to define the covering $\mathbb{C}_4$ and the fuzzy neighborhood operators $N_2^{\mathbb{C}_j}$ and $N_4^{\mathbb{C}_j}$ and $\mathcal{I}$ its R-implicator which is used to define $\mathbb{C}_3$ and the fuzzy neighborhood operators $N_1^{\mathbb{C}_j}$ and $N_3^{\mathbb{C}_j}$, in order to guarantee all equalities of Table 6.1.

We define a partial order relation $\preceq$ between fuzzy neighborhood operators as follows: let N and N′ be fuzzy neighborhood operators on U, then we write $\mathrm{N} \preceq \mathrm{N}'$ if and only if $\forall x, y \in U : \mathrm{N}(x)(y) \leq \mathrm{N}'(x)(y)$. We say that two fuzzy neighborhood operators N and N′ are incomparable with respect to $\preceq$ if neither $\mathrm{N} \preceq \mathrm{N}'$ nor $\mathrm{N}' \preceq \mathrm{N}$ hold. Note that if two crisp neighborhood operators are incomparable, their fuzzy extensions are incomparable as well, e.g., the fuzzy neighborhood operators $N_3^{\mathbb{C}_3}$ (group b) and $N_3^{\mathbb{C}_1}$ (group d) are incomparable. Therefore, we only need to consider the partial order relations given in Figure 4.1a. Moreover, let N and N′ be two fuzzy neighborhood operators of Table 6.1. If N and N′ are different, then $\mathrm{N} \preceq \mathrm{N}'$ implies that $\mathrm{N}' \preceq \mathrm{N}$ cannot hold.

In [141] it was proven that for each crisp covering $\mathbb{C}$ it holds that $N_1^{\mathbb{C}} \preceq N_2^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$ and $N_1^{\mathbb{C}} \preceq N_3^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$. These relationships are maintained in the fuzzy setting.

Proposition 6.5.1. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathcal{T}$ a left-continuous t-norm used to define the fuzzy neighborhood operators $N_2^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ and $\mathcal{I}$ its R-implicator used to define the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$, then

(a) $N_1^{\mathbb{C}} \preceq N_2^{\mathbb{C}}$,

(b) $N_1^{\mathbb{C}} \preceq N_3^{\mathbb{C}}$,

(c) $N_2^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$,

(d) $N_3^{\mathbb{C}} \preceq N_4^{\mathbb{C}}$.

Proof. (a) Assume that the inclusion does not hold for $x, y \in U$, then

$$\sup_{K\in \mathrm{md}(\mathbb{C},x)} \mathcal{T}(K(x), K(y)) < \inf_{K\in\mathbb{C}} \mathcal{I}(K(x), K(y)),$$

i.e., for all $K_1 \in \mathrm{md}(\mathbb{C}, x)$ and for all $K_2 \in \mathbb{C}$ it holds that

$$\mathcal{T}(K_1(x), K_1(y)) < \mathcal{I}(K_2(x), K_2(y)).$$

Take $K^* \in \mathbb{C}$ such that $K^*(x) = 1$ and take $K' \in \mathrm{md}(\mathbb{C}, x)$ such that

$$K'(x) = K^*(x) = 1$$

and $K' \subseteq K^*$. Then for $K_1 = K_2 = K'$ we have

$$\mathcal{T}(K'(x), K'(y)) < \mathcal{I}(K'(x), K'(y)),$$

hence, $K'(y) < K'(y)$, which is a contradiction.

(b) This follows immediately from the fact that $\mathrm{MD}(\mathbb{C}, x) \subseteq \mathbb{C}$.

(c) This follows immediately from the fact that $\mathrm{md}(\mathbb{C}, x) \subseteq \mathbb{C}$.

(d) Assume that the inclusion does not hold for $x, y \in U$, then

$$\sup_{K\in\mathbb{C}} \mathcal{T}(K(x), K(y)) < \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \mathcal{I}(K(x), K(y)),$$

i.e., for all $K_1 \in \mathbb{C}$ and for all $K_2 \in \mathrm{MD}(\mathbb{C}, x)$ it holds that

$$\mathcal{T}(K_1(x), K_1(y)) < \mathcal{I}(K_2(x), K_2(y)).$$

Take $K^* \in \mathbb{C}$ such that $K^*(x) = 1$ and take $K' \in \mathrm{MD}(\mathbb{C}, x)$ such that $K'(x) = K^*(x) = 1$ and $K^* \subseteq K'$. Then for $K_1 = K_2 = K'$ we have

$$\mathcal{T}(K'(x), K'(y)) < \mathcal{I}(K'(x), K'(y)),$$

hence, $K'(y) < K'(y)$, which is a contradiction.

□

Note that (a) and (d) of Proposition 6.5.1 uses Proposition 6.1.5, while the partial order relations in (b) and (c) also hold for an infinite $\mathbb{C}$. Proposition 6.5.1 implies that the following partial order relations hold for Table 6.1.

Corollary 6.5.2. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathcal{T}$ a left-continuous t-norm used to define the fuzzy covering $\mathbb{C}_4$ and the fuzzy neighborhood operators $N_2^{\mathbb{C}_j}$ and $N_4^{\mathbb{C}_j}$ and $\mathcal{I}$ its R-implicator used to define the fuzzy covering $\mathbb{C}_3$ and the fuzzy neighborhood operators $N_1^{\mathbb{C}_j}$ and $N_3^{\mathbb{C}_j}$, then in terms of the groups of Table 6.1 we obtain

(a) for $\mathbb{C}$: $a_1 \preceq c \preceq j_1$ and $a_1 \preceq f_1 \preceq j_1$,

(b) for $\mathbb{C}_1$: $a_1 \preceq c \preceq h$ and $a_1 \preceq d \preceq h$,

(c) for $\mathbb{C}_2$: $f_2 \preceq j_2 \preceq j_1$ and $f_2 \preceq f_1 \preceq j_1$,

(d) for $\mathbb{C}_3$: $a_1 \preceq a_2 \preceq g$ and $a_1 \preceq b \preceq g$,

(e) for $\mathbb{C}_4$: $i \preceq k \preceq m$ and $i \preceq l \preceq m$,

(f) for $\mathbb{C}_\cap$: $a_1 \preceq e \preceq j_1$ and $a_1 \preceq f_1 \preceq j_1$.

Moreover, since $\mathbb{C}_1$ and $\mathbb{C}_2$ are subcoverings of the finite fuzzy covering $\mathbb{C}$, we obtain that $a_1 \preceq f_2$ and $h \preceq j_1$.

Proposition 6.5.3. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathcal{T}$ a left-continuous t-norm used to define the fuzzy neighborhood operators $N_4^{\mathbb{C}}$ and $N_4^{\mathbb{C}_1}$ and $\mathcal{I}$ its R-implicator used to define the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_1^{\mathbb{C}_1}$, then $N_1^{\mathbb{C}} \preceq N_1^{\mathbb{C}_2}$ and $N_4^{\mathbb{C}_1} \preceq N_4^{\mathbb{C}}$.

Proof. This follows immediately from $\mathbb{C}_2 \subseteq \mathbb{C}$ and $\mathbb{C}_1 \subseteq \mathbb{C}$. □

Furthermore, it holds that $f_1 \preceq i \preceq j_1 \preceq k$. To prove these partial order relations, we first consider the following lemma.

Lemma 6.5.4. [138] Let $\mathcal{T}$ a left-continuous t-norm and $\mathcal{I}$ its R-implicator, then $\mathcal{I}(a,b) \leq \mathcal{I}(\mathcal{T}(a,c), \mathcal{T}(c,b))$ holds for all $a, b, c \in [0,1]$.

Proposition 6.5.5. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathscr{T}$ a left-continuous t-norm used to define the fuzzy covering $\mathbb{C}_4$ and the fuzzy neighborhood operators $N_4^{\mathbb{C}}$ and $N_2^{\mathbb{C}_4}$ and $\mathscr{I}$ its R-implicator used to define the fuzzy neighborhood operators $N_3^{\mathbb{C}}$ and $N_1^{\mathbb{C}_4}$, then

(a) $N_3^{\mathbb{C}} \preceq N_1^{\mathbb{C}_4}$,

(b) $N_1^{\mathbb{C}_4} \preceq N_4^{\mathbb{C}}$,

(c) $N_4^{\mathbb{C}} \preceq N_2^{\mathbb{C}_4}$.

Proof. (a) Let $x, y \in U$, then

$$\begin{aligned}
N_1^{\mathbb{C}_4}(x)(y) &= \inf_{z\in U} \mathscr{I}(N_4^{\mathbb{C}}(z)(x), N_4^{\mathbb{C}}(z)(y)) \\
&= \inf_{z\in U} \mathscr{I}(N_4^{\mathbb{C}}(x)(z), N_4^{\mathbb{C}}(z)(y)) \\
&= \inf_{z\in U} \mathscr{I}\left(\sup_{K\in \mathrm{MD}(\mathbb{C},x)} \mathscr{T}(K(x),K(z)), N_4^{\mathbb{C}}(z)(y)\right) \\
&= \inf_{z\in U} \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \mathscr{I}\left(\mathscr{T}(K(x),K(z)), \sup_{K'\in\mathbb{C}} \mathscr{T}(K'(z),K'(y))\right) \\
&\geq \inf_{z\in U} \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \sup_{K'\in\mathbb{C}} \mathscr{I}\left(\mathscr{T}(K(x),K(z)), \mathscr{T}(K'(z),K'(y))\right) \\
&\geq \inf_{z\in U} \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \mathscr{I}\left(\mathscr{T}(K(x),K(z)), \mathscr{T}(K(z),K(y))\right) \\
&= \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \inf_{z\in U} \mathscr{I}\left(\mathscr{T}(K(x),K(z)), \mathscr{T}(K(z),K(y))\right) \\
&\geq \inf_{K\in \mathrm{MD}(\mathbb{C},x)} \mathscr{I}(K(x),K(y)) \\
&= N_3^{\mathbb{C}}(x)(y)
\end{aligned}$$

where in the penultimate step we have used Lemma 6.5.4.

(b) Let $x, y \in U$, then

$$\begin{aligned}
N_1^{\mathbb{C}_4}(x)(y) &\leq \mathscr{I}(N_4^{\mathbb{C}}(x)(x), N_4^{\mathbb{C}}(x)(y)) \\
&= \mathscr{I}(1, N_4^{\mathbb{C}}(x)(y)) \\
&= N_4^{\mathbb{C}}(x)(y).
\end{aligned}$$

(c) Let $x, y \in U$. If $N_4^{\mathbb{C}}(x) \in \text{md}(\mathbb{C}_4, x)$, then we have that

$$N_2^{\mathbb{C}_4}(x)(y) \geq \mathscr{T}(N_4^{\mathbb{C}}(x)(x), N_4^{\mathbb{C}}(x)(y)) = \mathscr{T}(1, N_4^{\mathbb{C}}(x)(y)) = N_4^{\mathbb{C}}(x)(y).$$

Similarly, if $N_4^{\mathbb{C}}(y) \in \text{md}(\mathbb{C}_4, x)$, then

$$N_2^{\mathbb{C}_4}(x)(y) \geq \mathscr{T}(N_4^{\mathbb{C}}(y)(x), N_4^{\mathbb{C}}(y)(y)) = N_4^{\mathbb{C}}(x)(y).$$

Now assume that neither $N_4^{\mathbb{C}}(x)$ nor $N_4^{\mathbb{C}}(y)$ belong to $\text{md}(\mathbb{C}_4, x)$. Hence,

$$\begin{array}{l} \exists z_1 \in U : N_4^{\mathbb{C}}(z_1)(x) = N_4^{\mathbb{C}}(x)(x) = 1, N_4^{\mathbb{C}}(z_1) \subseteq N_4^{\mathbb{C}}(x), \\ \exists z_2 \in U : N_4^{\mathbb{C}}(z_2)(x) = N_4^{\mathbb{C}}(y)(x), N_4^{\mathbb{C}}(z_2) \subseteq N_4^{\mathbb{C}}(y), \end{array}$$

with $N_4^{\mathbb{C}}(z_1), N_4^{\mathbb{C}}(z_2) \in \text{md}(\mathbb{C}_4, x)$. Note that

$$N_4^{\mathbb{C}}(y)(z_2) \geq N_4^{\mathbb{C}}(z_2)(z_2) = 1.$$

We derive that

$$\begin{array}{rl} & N_2^{\mathbb{C}_4}(x)(y) \\ \geq & \max\big(\mathscr{T}(N_4^{\mathbb{C}}(z_1)(x), N_4^{\mathbb{C}}(z_1)(y)), \mathscr{T}(N_4^{\mathbb{C}}(z_2)(x), N_4^{\mathbb{C}}(z_2)(y))\big) \\ = & \max\big(\mathscr{T}(1, N_4^{\mathbb{C}}(z_1)(y)), \mathscr{T}(N_4^{\mathbb{C}}(y)(x), 1)\big) \\ = & \max\big(N_4^{\mathbb{C}}(z_1)(y), N_4^{\mathbb{C}}(x)(y)\big) \\ = & N_4^{\mathbb{C}}(x)(y). \end{array}$$

□

Furthermore, we can prove that $g \preceq h$.

Proposition 6.5.6. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathscr{T}$ a left-continuous t-norm used to define the fuzzy neighborhood operators $N_4^{\mathbb{C}_3}$ and $N_4^{\mathbb{C}_1}$ and $\mathscr{I}$ its R-implicator used to define the fuzzy covering $\mathbb{C}_3$, then $N_4^{\mathbb{C}_3} \preceq N_4^{\mathbb{C}_1}$.

Proof. Let $x, y, z \in U$ and let $K^* \in \text{md}(\mathbb{C}, z)$ with $K^*(z) = 1$, then

$$\mathscr{T}(N_1^{\mathbb{C}}(z)(x), N_1^{\mathbb{C}}(z)(y)) \quad \leq \quad \inf_{K \in \text{md}(\mathbb{C}, z)} \mathscr{T}\left(\mathscr{I}(K(z), K(x)), \mathscr{I}(K(z), K(y))\right)$$

$$
\begin{aligned}
&\leq \mathcal{T}(\mathcal{I}(K^*(z),K^*(x)),\mathcal{I}(K^*(z),K^*(y)))\\
&= \mathcal{T}(\mathcal{I}(1,K^*(x)),\mathcal{I}(1,K^*(y)))\\
&= \mathcal{T}(K^*(x),K^*(y))\\
&\leq \sup_{K\in \mathrm{md}(\mathbb{C},z)} \mathcal{T}(K(x),K(y)).
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
N_4^{\mathbb{C}_3}(x)(y) &= \sup_{z\in U}\mathcal{T}(N_1^{\mathbb{C}}(z)(x),N_1^{\mathbb{C}}(z)(y))\\
&\leq \sup_{z\in U}\sup_{K\in \mathrm{md}(\mathbb{C},z)} \mathcal{T}(K(x),K(y))\\
&= \sup_{K\in\mathbb{C}_1}\mathcal{T}(K(x),K(y))\\
&= N_4^{\mathbb{C}_1}(x)(y).
\end{aligned}
$$

□

To end, we show that $e \preceq j_2$.

Proposition 6.5.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite and $\mathcal{T}$ a left-continuous t-norm used to define the fuzzy neighborhood operators $N_2^{\mathbb{C}_\cap}$ and $N_2^{\mathbb{C}_2}$, then $N_2^{\mathbb{C}_\cap} \preceq N_2^{\mathbb{C}_2}$.

Proof. Let $x \in U$, then we first prove that $\mathrm{md}(\mathbb{C}_\cap,x)\cap\mathbb{C}_2 \subseteq \mathrm{md}(\mathbb{C}_2,x)$. Let $K\in \mathrm{md}(\mathbb{C}_\cap,x)\cap\mathbb{C}_2$ and take $K'\in\mathbb{C}_2$ with $K'(x)=K(x)$ and $K'\subseteq K$. As $K'\in\mathbb{C}_\cap$ and $K\in\mathrm{md}(\mathbb{C}_\cap,x)$, we obtain $K=K'$ and thus, $K\in\mathrm{md}(\mathbb{C}_2,x)$.

Let $y\in U$ and assume that $N_2^{\mathbb{C}_\cap}(x)(y)=\mathcal{T}(K^*(x),K^*(y))$ for $K^*\in\mathrm{md}(\mathbb{C}_\cap,x)$. If $K^*\notin\mathbb{C}_2$, then for each $z\in U$ there exists a $K_z\in\mathbb{C}_2$ such that $K^*(z)=K_z(z)$ and $K^*\subsetneq K_z$. However, this means that $K^*=\bigcap_{z\in U}K_z$. As $K^*\in\mathbb{C}_\cap$, this is a contradiction. Therefore, K^* will be a set in $\mathbb{C}_2$ and by the observation above, $K^*\in\mathrm{md}(\mathbb{C}_2,x)$ is obtained and thus,

$$N_2^{\mathbb{C}_\cap}(x)(y)\leq N_2^{\mathbb{C}_2}(x)(y).$$

□

By the transitivity of $\preceq$, Figure 4.1a and the results obtained above, the only partial order relations which also can hold are the following:

(a) $a_2 \preceq \mathrm{N}$ with N a fuzzy neighborhood operator of groups b, c, d, e, f_1, f_2, i, j_2 and l,

(b) $\mathrm{N} \preceq j_2$ with N a fuzzy neighborhood operator of groups b, c, d, f_1, g, h and i.

However, none of these partial order relations hold, as illustrated in the following two examples.

Example 6.5.8. Let $U = \{x, y, z\}$, $\mathcal{T}$ the product t-norm and $\mathcal{I}$ its R-implicator. Let $\mathbb{C} = \{K_1, K_2, K_3, K_4\}$ with $K_1 = 1/x + 0.2/y + 0.6/z$, $K_2 = 0.5/x + 1/y + 0.6/z$, $K_3 = 0.5/x + 0.6/y + 1/z$ and $K_4 = 0.5/x + 0.5/y + 0/z$. Then $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}_\cap = \mathbb{C}$ and

$$\mathbb{C}_3 = \{1/x + 0.2/y + 0/z, 0.5/x + 1/y + 0/z, 0.5/x + \frac{1}{3}/y + 1/z\}.$$

Hence, we have that

- $N_2^{\mathbb{C}_3}(x)(y) = 0.5$ (group a_2),
- $N_2^{\mathbb{C}}(x)(y) = N_2^{\mathbb{C}_\cap}(x)(y) = N_2^{\mathbb{C}_2}(x)(y) = 0.25$ (groups c, e and j_2),
- $N_3^{\mathbb{C}_3}(x)(y) = N_3^{\mathbb{C}_1}(x)(y) = N_3^{\mathbb{C}}(x)(y) = N_1^{\mathbb{C}_2}(x)(y) = 0.2$ (groups b, d, f_1 and f_2).

Hence, $N_2^{\mathbb{C}_3} \preceq \mathrm{N}$ does not hold for $\mathrm{N} \in \{N_3^{\mathbb{C}_3}, N_2^{\mathbb{C}}, N_3^{\mathbb{C}_1}, N_2^{\mathbb{C}_\cap}, N_3^{\mathbb{C}}, N_1^{\mathbb{C}_2}, N_2^{\mathbb{C}_2}\}$. Therefore, the fuzzy neighborhood operator $N_2^{\mathbb{C}_3}$ is incomparable with the fuzzy neighborhood operators of the groups $b - f_2$ and j_2.

On the other hand, let $\mathbb{C} = \{K_1, K_2, K_3, K_4\}$ with $K_1 = 1/x + 0.8/y + 0.6/z$, $K_2 = 0.2/x + 1/y + 0.6/z$, $K_3 = 0.2/x + 0.8/y + 1/z$ and $K_4 = 0.1/x + 0.6/y + 1/z$ and let $\mathcal{T}$ be the minimum operator and $\mathcal{I}$ its R-implicator. Then

$$\mathbb{C}_3 = \{1/x + 0.8/y + 0.6/z, 0.1/x + 1/y + 0.6/z, 0.1/x + 0.6/y + 1/z\}$$

and

$$\mathbb{C}_4 = \{1/x + 0.8/y + 0.6/z, 0.8/x + 1/y + 0.8/z, 0.6/x + 0.8/y + 1/z\}.$$

Hence, we have that $N_2^{\mathbb{C}_3}(y)(x) = 0.8$ and $N_1^{\mathbb{C}_4}(y)(x) = N_3^{\mathbb{C}_4}(y)(x) = 0.6$, thus, $N_2^{\mathbb{C}_3} \preceq N_1^{\mathbb{C}_4}$ and $N_2^{\mathbb{C}_3} \preceq N_3^{\mathbb{C}_4}$ do not hold. Therefore, the fuzzy neighborhood operator $N_2^{\mathbb{C}_3}$ is incomparable with the fuzzy neighborhood operators of the groups i and l.

Example 6.5.9. Let $U = \{x, y, z\}$ with $\mathbb{C}$ as in Example 6.3.2, $\mathscr{T}$ the minimum operator and $\mathscr{I}$ its R-implicator, then $N_2^{\mathbb{C}_2}(x)(y) = 0.1$ and

$$N_2^{\mathbb{C}}(x)(y) = N_3^{\mathbb{C}_1}(x)(y) = N_3^{\mathbb{C}}(x)(y) = N_4^{\mathbb{C}_3}(x)(y) = N_1^{\mathbb{C}_4}(x)(y) = 0.9.$$

Hence, $\mathrm{N} \preceq N_2^{\mathbb{C}_2}$ does not hold for $\mathrm{N} \in \{N_2^{\mathbb{C}}, N_3^{\mathbb{C}_1}, N_3^{\mathbb{C}}, N_4^{\mathbb{C}_3}, N_1^{\mathbb{C}_4}\}$. Therefore, the fuzzy neighborhood operator $N_2^{\mathbb{C}_2}$ is incomparable with the fuzzy neighborhood operators of the groups c, d, f_1, g and i.

On the other hand, let $\mathbb{C} = \{K_1, K_2, K_3, K_4\}$ with $K_1 = 1/x + 0.2/y + 0.6/z$, $K_2 = 0.5/x + 1/y + 0.6/z$, $K_3 = 0.5/x + 0.6/y + 1/z$ and $K_4 = 0.5/x + 0.5/y + 0/z$ and $\mathscr{T}$ the product t-norm and $\mathscr{I}$ its R-implicator as in Example 6.5.8, then we obtain that $N_2^{\mathbb{C}_2}(x)(y) = 0.25$ and $N_4^{\mathbb{C}_1}(x)(y) = 0.5$. Hence, $N_4^{\mathbb{C}_1} \preceq N_2^{\mathbb{C}_2}$ does not hold. Therefore, the fuzzy neighborhood operator $N_2^{\mathbb{C}_2}$ is incomparable with the fuzzy neighborhood operator $N_4^{\mathbb{C}_1}$ (group h).

To end, we illustrate that the fuzzy neighborhood operator $N_2^{\mathbb{C}_2}$ is incomparable with the fuzzy neighborhood operator $N_3^{\mathbb{C}_3}$ (group b). Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ a fuzzy covering on U with $K_1 = 1/x + 0.5/y$ and $K_2 = 1/x + 1/y$ and let $\mathscr{T}$ be the minimum operator and $\mathscr{I}$ its R-implicator, then $\mathbb{C}_2 = \mathbb{C}_3 = \mathbb{C}$. Hence, $N_2^{\mathbb{C}_2}(x)(y) = 0.5$ while $N_3^{\mathbb{C}_3}(x)(y) = 1$. Hence, $N_3^{\mathbb{C}_3} \preceq N_2^{\mathbb{C}_2}$ does not hold.

The Hasse diagram with respect to $\preceq$ for the fuzzy neighborhood operators presented in Table 6.1 is given in Figure 6.1. Note that the Hasse diagram is a lattice, with minimum a_1 and maximum m.

To end this section, we discuss the properties of the fuzzy neighborhood operators presented in Table 6.1. By Proposition 6.2.8 it holds that all fuzzy neighborhood operators in Table 6.1 are reflexive when a finite fuzzy covering, a left-continuous t-norm and its R-implicator are considered.

From the discussion in Section 4.1.1 we know that for a crisp covering $\mathbb{C}$ it holds that the neighborhood operators of groups g, h, j and m are symmetric and the neighborhood operators of groups a, b, d, f, i and l are transitive. We study whether these properties are maintained in the fuzzy setting.

First, we prove that the fuzzy neighborhood operator $N_4^{\mathbb{C}}$ is symmetric.

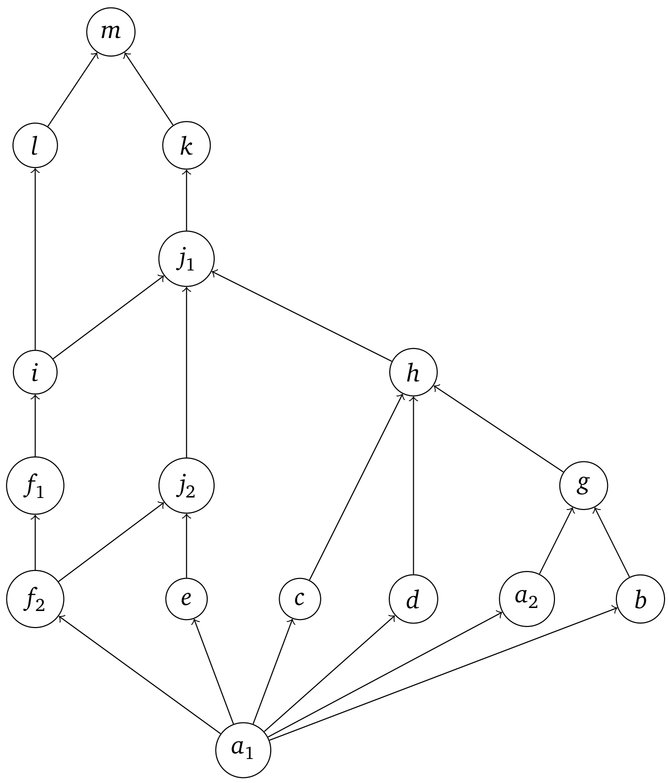


Figure 6.1: Hasse diagram of the fuzzy neighborhood operators from Table 6.1 for $(U, \mathbb{C})$ with $\mathbb{C}$ finite, $\mathscr{T}$ a left-continuous t-norm and $\mathscr{I}$ its R-implicator

Proposition 6.5.10. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{T}$ a t-norm, then the fuzzy neighborhood operator $N_4^{\mathbb{C}}$ based on $\mathscr{T}$ is symmetric.

Proof. This follows immediately since a t-norm is commutative. □

From this, we obtain that the fuzzy neighborhood operators of groups g, h, j_1 and m are symmetric. The fuzzy neighborhood operator $N_2^{\mathbb{C}_2}$ (group j_2) is not symmetric as illustrated in the next example:

Example 6.5.11. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.5/y$ and $K_2 = 1/x + 1/y$, then $\mathbb{C}_2 = \mathbb{C}$. Let $\mathscr{T}$ be a t-norm, then

- $N_2^{\mathbb{C}_2}(x)(y) = \mathscr{T}(K_1(x), K_1(y)) = 0.5$,
- $N_2^{\mathbb{C}_2}(y)(x) = \max(\mathscr{T}(K_1(y), K_1(x)), \mathscr{T}(K_2(y), K_2(x))) = 1$.

Hence, we conclude that $N_2^{\mathbb{C}_2}$ is not symmetric.

Second, the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are $\mathscr{T}$-transitive for a left-continuous t-norm $\mathscr{T}$ if the used implicator is the R-implicator of $\mathscr{T}$. In order to prove transitivity, we first consider the following result:

Lemma 6.5.12. [138] Let $\mathscr{T}$ be a left-continuous t-norm and $\mathscr{I}$ its R-implicator, then $\mathscr{T}(\mathscr{I}(a,b),\mathscr{I}(b,c)) \leq \mathscr{I}(a,c)$ for all $a,b,c \in [0,1]$.

Based on this lemma, we can prove the transitivity of the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$.

Proposition 6.5.13. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space and let $\mathscr{T}$ be a left-continuous t-norm and $\mathscr{I}$ its R-implicator used in the definition of the fuzzy neighborhood operators $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$, then $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are $\mathscr{T}$-transitive fuzzy neighborhood operators.

Proof. We have for each $x, y, z \in U$ that

$$\begin{aligned}
&\mathscr{T}\left(N_1^{\mathbb{C}}(x)(y), N_1^{\mathbb{C}}(y)(z)\right)\\
&\quad = \mathscr{T}\left(\inf_{K\in\mathbb{C}} \mathscr{I}(K(x),K(y)), \inf_{K\in\mathbb{C}} \mathscr{I}(K(y),K(z))\right)\\
&\quad \leq \inf_{K\in\mathbb{C}} \mathscr{T}\left(\mathscr{I}(K(x),K(y)), \mathscr{I}(K(y),K(z))\right)\\
&\quad \leq \inf_{K\in\mathbb{C}} \mathscr{I}(K(x),K(z))\\
&\quad = N_1^{\mathbb{C}}(x)(z),
\end{aligned}$$

where in the penultimate step Lemma 6.5.12 is used. The proof for $N_3^{\mathbb{C}}$ is similar. □

From this proposition, we immediately obtain that the fuzzy neighborhood operators of groups a_1, b, d, f_1, f_2, i and l are $\mathscr{T}$-transitive, when a left-continuous t-norm and its R-implicator is considered. The fuzzy neighborhood operator $N_2^{\mathbb{C}_3}$ (group a_2) is not $\mathscr{T}$-transitive, as illustrated in the next example for $\mathscr{T} = \mathscr{T}_M$:

Example 6.5.14. Let $U = \{x, y, z\}$ and $\mathbb{C} = \{K_1, K_2, K_3\}$ a fuzzy covering on U with $K_1 = 1/x + 0.8/y + 0/z$, $K_2 = 0.8/x + 1/y + 1/z$ and $K_3 = 0.5/x + 0.6/y + 1/z$. Let $\mathscr{T}$ be the minimum operator and $\mathscr{I}$ its R-implicator. We obtain with this implicator that

$$\mathbb{C}_3 = \{1/x + 0.8/y + 0/z, 0.5/x + 1/y + 0/z, 0.5/x + 0.6/y + 1/z\}.$$

Thus, we derive that $N_2^{\mathbb{C}_3}(x)(y) = 0.8$, $N_2^{\mathbb{C}_3}(y)(z) = 0.6$ and $N_2^{\mathbb{C}_3}(x)(z) = 0.5$, hence, $0.6 = \mathscr{T}(0.8, 0.6) > 0.5$. We conclude that $N_2^{\mathbb{C}_3}$ is not $\mathscr{T}_M$-transitive.

Note that in the proof of Proposition 6.5.13 we have explicitly used properties specific to a left-continuous t-norm and its R-implicator. We illustrate this with an example.

Example 6.5.15. Let $U = \{x, y, z\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.1/y + 1/z$ and $K_2 = 0.5/x + 1/y + 0.2/z$. Let $\mathscr{T}$ be the minimum operator $\mathscr{T}_M$ and $\mathscr{I}$ the Kleene-Dienes implicator which is used to define $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$. Since for all $x \in U$ it holds that $\mathrm{MD}(\mathbb{C}, x) = \mathbb{C}$, we have that $N_3^{\mathbb{C}} = N_1^{\mathbb{C}}$. However, since $N_1^{\mathbb{C}}(y)(x) = N_1^{\mathbb{C}}(x)(z) = 0.5$ and $N_1^{\mathbb{C}}(y)(z) = 0.2$, we obtain that $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ defined with the Kleene-Dienes implicator are not $\mathscr{T}_M$-transitive.

6.6 Fuzzy neighborhood operator introduced by Ma

In [107], Ma introduced a family of fuzzy neighborhood operators based on a fuzzy covering. Let $\beta \in (0,1]$, then the *β-fuzzy neighborhood operator* is defined as follows

$$N^{\mathbb{C}}_{\beta,\mathrm{Ma}} \colon U \to \mathscr{F}(U) \colon x \mapsto N^{\mathbb{C}}_{\beta,\mathrm{Ma}}(x) \tag{6.19}$$

such that the fuzzy neighborhood $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}(x)$ of the object $x \in U$ is defined by

$$\forall y \in U \colon N^{\mathbb{C}}_{\beta,\mathrm{Ma}}(x)(y) = \inf\{K(y) \mid K \in \mathbb{C}, K(x) \geq \beta\}. \tag{6.20}$$

Note that Feng et al. described this fuzzy neighborhood operator for $\beta = 1$ in [45]. The parameter β is strictly greater than 0 for the following reason: the fuzzy neighborhood operator $N^{\mathbb{C}}_{0,\mathrm{Ma}}$ would map each object x to the fuzzy set $X^{\mathbb{C}}_{\inf}$ with

$$\forall y \in U \colon X^{\mathbb{C}}_{\inf}(y) = \inf\{K(y) \mid K \in \mathbb{C}\}.$$

Next, we study the properties of the fuzzy neighborhood operator $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}$ for $\beta \in (0,1]$. Assume $\beta < 1$, then the fuzzy neighborhood operator $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}$ is not necessarily reflexive as illustrated in the next example:

Example 6.6.1. Assume $\beta \in (0,1)$. Let $U = \{x\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1(x) = 1$ and $K_2(x) = \beta$, then $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}(x)(x) = \beta$, i.e., $N^{\mathbb{C}}_{\beta,\mathrm{Ma}}$ is not reflexive.

However, the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ is reflexive:

Proposition 6.6.2. [45] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, then $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ is a reflexive fuzzy neighborhood operator.

By this result, we will only consider the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\mathrm{Ma}}$. Note that this fuzzy neighborhood operator is neither symmetric nor $\mathscr{T}$-transitive:

Example 6.6.3. Let $U = \{x, y, z\}$ and $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 1/x + 0.8/y + 0/z$, $K_2 = 0.8/x + 1/y + 1/z$ and $K_3 = 0.5/x + 0.6/y + 1/z$. Then we have the following observations:

(a) $N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(z) = 0$ and $N^{\mathbb{C}}_{1,\mathrm{Ma}}(z)(x) = 0.5$, hence, $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ is not symmetric.

(b) $N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(y) = 0.8$, $N^{\mathbb{C}}_{1,\mathrm{Ma}}(y)(z) = 1$ and $N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(z) = 0$, then for every t-norm $\mathcal{T}$ it holds that $0.8 = \mathcal{T}(0.8, 1) > 0.5$. Hence, $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ is not $\mathcal{T}$-transitive for any t-norm $\mathcal{T}$.

We now want to add the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\mathrm{Ma}}$ to the Hasse diagram presented in Figure 6.1. We have the following partial order relations with respect to $\preceq$:

Proposition 6.6.4. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with $\mathbb{C}$ finite, $\mathcal{T}$ a left-continous t-norm and $\mathcal{I}$ its R-implicator, then

(a) $N^{\mathbb{C}}_1 \preceq N^{\mathbb{C}}_{1,\mathrm{Ma}}$,

(b) $N^{\mathbb{C}}_{1,\mathrm{Ma}} \preceq N^{\mathbb{C}}_2$,

(c) $N^{\mathbb{C}}_{1,\mathrm{Ma}} \preceq N^{\mathbb{C}_\cap}_2$.

Proof. (a) Let $x, y \in U$.

$$\begin{aligned}
N^{\mathbb{C}}_1(x)(y) &= \inf_{K\in\mathbb{C}} \mathcal{I}(K(x), K(y)) \\
&\leq \inf_{K\in\mathbb{C}, K(x)=1} \mathcal{I}(K(x), K(y)) \\
&= \inf_{K\in\mathbb{C}, K(x)=1} K(y) \\
&= N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(y).
\end{aligned}$$

(b) Let $x, y \in U$.

$$\begin{aligned}
N^{\mathbb{C}}_2(x)(y) &= \sup_{K\in\mathrm{md}(\mathbb{C},x)} \mathcal{T}(K(x), K(y)) \\
&\geq \sup_{K\in\mathrm{md}(\mathbb{C},x), K(x)=1} \mathcal{T}(K(x), K(y)) \\
&= \sup_{K\in\mathrm{md}(\mathbb{C},x), K(x)=1} K(y) \\
&\geq \inf_{K\in\mathrm{md}(\mathbb{C},x), K(x)=1} K(y) \\
&\geq \inf_{K\in\mathbb{C}, K(x)=1} K(y) \\
&= N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(y)
\end{aligned}$$

(c) By definition of $\mathbb{C}_\cap$, we have that

$$N^{\mathbb{C}}_{1,\text{Ma}}(x)(y) = \inf\{K(y) \mid K \in \mathbb{C}_\cap, K(x) = 1\}.$$

By analogy of (b), we obtain that

$$N_2^{\mathbb{C}_\cap}(x)(y) \geq \inf_{K \in \mathbb{C}_\cap, K(x)=1} K(y) = N^{\mathbb{C}}_{1,\text{Ma}}(x)(y).$$

□

Note that the previous proposition also holds for every t-norm $\mathcal{T}$ and every border implicator $\mathcal{I}$.

As $\preceq$ is a transitive relation, we also have $N^{\mathbb{C}}_{1,\text{Ma}} \preceq \text{N}$ for N a fuzzy neighborhood operator of groups h, j_1, j_2, k, m. There are no other partial order relations as illustrated in the next example.

Example 6.6.5. Let $U = \{x, y, z\}$, $\mathcal{T}$ the Łukasiewicz t-norm and $\mathcal{I}$ its R-implicator. Let $\mathbb{C} = \{K_1, K_2, K_3, K_4, K_5\}$ with $K_1 = 1/x + 0.5/y + 0.8/z$, $K_2 = 1/x + 0.8/y + 0.5/z$, $K_3 = 1/x + 0.8/y + 1/z$, $K_4 = 0.5/x + 0.8/y + 0.5/z$ and $K_5 = 0.5/x + 1/y + 0.5/z$. Then $\mathbb{C}_1 = \mathbb{C}$, $\mathbb{C}_2 = \mathbb{C}_\cap = \{K_1, K_2, K_3, K_5\}$,

$$\mathbb{C}_3 = \{1/x + 0.5/y + 0.5/z, 0.5/x + 1/y + 0.5/z, 1/x + 0.7/y + 1/z\}$$

and

$$\mathbb{C}_4 = \{1/x + 0.8/y + 1/z, 0.8/x + 1/y + 0.8/z, 1/x + 0.8/y + 1/z\}.$$

Hence, we derive that

- $N^{\mathbb{C}}_{1,\text{Ma}}(x)(y) = 0.5 < 0.8 = \text{N}(x)(y)$ for N a fuzzy neighborhood operator of groups c, d, e, f_1, h, i, j_1, j_2, k, l and m,
- $N^{\mathbb{C}}_{1,\text{Ma}}(x)(y) = 0.5 < 0.7 = N_3^{\mathbb{C}_3}(x)(y) = N_4^{\mathbb{C}_3}(x)(y)$ (groups b and h),
- $N^{\mathbb{C}}_{1,\text{Ma}}(y)(x) = 0.5 < 0.7 = N_2^{\mathbb{C}_3}(y)(x)$ (group a_2),
- $N^{\mathbb{C}}_{1,\text{Ma}}(z)(y) = 0.8 > 0.7 = \text{N}(z)(y)$ for N a fuzzy neighborhood operator of groups a_1, a_2, b, d, f_1, f_2 and g.

On the other hand, consider $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 1/y + 0/z$ and $K_2 = 0.8/x + 0/y + 1/z$, then

$$\mathbb{C}_4 = \{1/x + 1/y + 0.8/z, 1/x + 1/y + 0/z, 0.8/x + 0/y + 1/z\}.$$

It holds that $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = 1 > 0.2 = N_1^{\mathbb{C}_4}(x)(y) = N_3^{\mathbb{C}_4}(x)(y)$ (groups i and l). Finally, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0/z$ and $K_2 = 1/x + 1/y + 1/z$, then $\mathbb{C}_2 = \{K_2\}$ and $N_{1,\mathrm{Ma}}^{\mathbb{C}}(x)(y) = 0 < 1 = N_1^{\mathbb{C}_2}(x)(y)$ (group f_2).

Hence, the Hasse diagram with respect to $\preceq$ of the 17 fuzzy neighborhood operators is given in Figure 6.2.

6.7 Conclusions

In this chapter we have introduced the definition of a fuzzy covering. Moreover, given a fuzzy covering we have introduced fuzzy extensions of the neighborhood system, the minimal description and maximal description of an object of the universe. In addition, four crisp neighborhood operators and six crisp coverings studied in [189] are extended to the fuzzy setting, and moreover, some results concerning crisp neighborhood operators and crisp coverings are maintained. For a finite fuzzy covering, the four fuzzy neighborhood operators and six fuzzy coverings, one original and five derived ones, result in 24 combinations of fuzzy neighborhood operators. However, we have proven that for a left-continuous t-norm and its residual implicator the obtained 24 combinations can be reduced to 16 different groups of fuzzy neighborhood operators. In this setting, we have derived the Hasse diagram of these 16 groups, which expresses which operators yield larger or smaller fuzzy neighborhoods. Finally, we have discussed a family of fuzzy neighborhood operators introduced by Ma in [107]. We have shown that only for the parameter $\beta = 1$ the fuzzy neighborhood operator is reflexive. We have discussed the properties of this fuzzy neighborhood operator and studied the partial order relations with the 16 fuzzy neighborhood operators. The Hasse diagram of these 17 fuzzy neighborhood operators can be found in Figure 6.2.

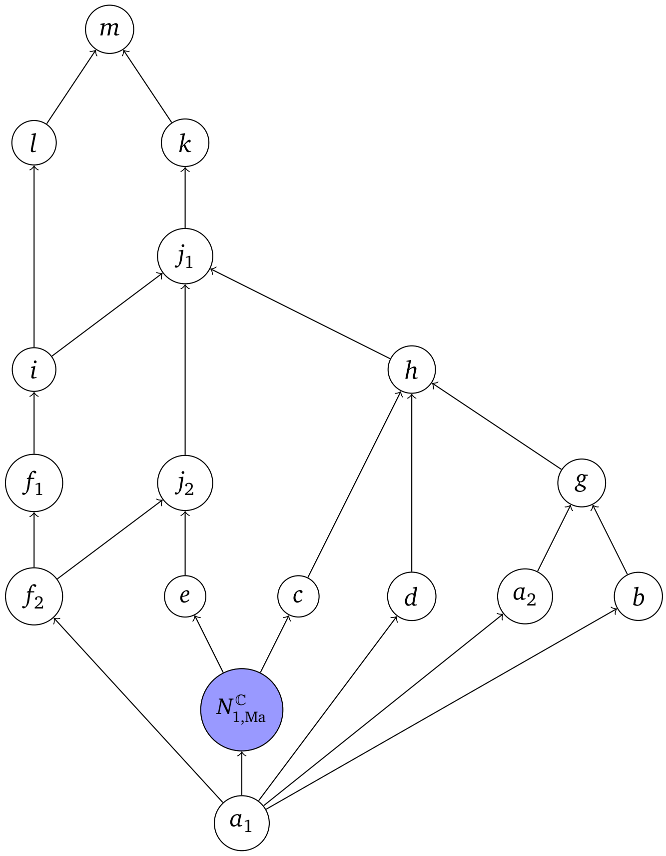


Figure 6.2: Hasse diagram of the fuzzy neighborhood operators for $(U,\mathbb{C})$ with $\mathbb{C}$ finite, $\mathscr{T}$ a left-continuous t-norm and $\mathscr{I}$ its R-implicator, where we have added the fuzzy neighborhood operator $N^{\mathbb{C}}_{1,\text{Ma}}$ to the Hasse diagram presented in Figure 6.1

CHAPTER 7

The implicator-conjunctor-based fuzzy rough set model

In this chapter, we first present a historical overview on the research of fuzzy rough set theory throughout the years. In Section 7.2, we introduce a general fuzzy rough set model which uses a fuzzy relation or fuzzy neighborhood operator to describe the indiscernibility between objects of the universe. This model extends the original rough set model of Pawlak to the fuzzy setting and it encapsulates many fuzzy rough set models discussed in literature. We discuss which properties the model satisfies when a fuzzy neighborhood operator is considered in Section 7.3. To end, we state conclusions in Section 7.4.

7.1 An introduction to fuzzy rough set theory

A drawback of rough set theory is its limitations when dealing with real-valued data. Fuzzy set theory [193] is very useful to overcome these limitations. It was recognized early on that rough set theory and fuzzy set theory are complementary, rather than competitive. To that end, rough set theory has been extended in two ways [41]. Rough fuzzy set theory discusses the approximation of a fuzzy set by a crisp indiscernibility relation. If moreover the indiscernibility relation is fuzzy as well, fuzzy rough set theory is considered. Since every crisp relation can be seen as a special case of a fuzzy relation, all results obtained in fuzzy rough set theory also hold for rough fuzzy set theory.

The vestiges of fuzzy rough set theory date back to the late 1980s, and originate from work by Fariñas del Cerro and Prade [44], Dubois and Prade [40], Nakamura [120] and Wygralak [172]. From 1990 onwards, research on the hybridization between rough sets and fuzzy sets has flourished. The inspiration to combine rough and fuzzy set theory was found in different mathematical fields. For instance, Lin [92] studied fuzzy rough sets using generalized topological spaces (Fréchet spaces) and Nanda and Majumdar [122] discussed fuzzy rough sets based on an algebraic approach. Moreover, Thiele [155] examined the relationship with fuzzy modal logic. Later on, Yao [180] and Liu [101] used level sets to combine fuzzy and rough set theory.

In this chapter, we will focus on fuzzy rough set models using fuzzy relations or fuzzy neighborhood operators. The seminal papers of Dubois and Prade [41,42] are probably the most important in the evolution of these fuzzy rough set models, since they influenced numerous authors who used different fuzzy logical connectives and fuzzy relations. Essential work was done by Morsi and Yakout in [118] who studied both constructive and axiomatic approaches and by Radzikowska and Kerre [137] who defined fuzzy rough sets based on three general classes of fuzzy implicators: S-, R- and QL-implicators. However, despite generalizing the fuzzy connectives, they still used fuzzy similarity relations. A first attempt to use reflexive fuzzy relations instead of fuzzy similarity relations was done by Greco et al. [51,52]. Thereafter, Wu et al. [170, 171] were the first to consider general fuzzy relations. Besides generalizing the fuzzy relation, Mi et al. [112, 113] considered conjunctors instead of t-norms. Furthermore, Yeung et al. [192] discussed two pairs of dual approximation operators from both a constructive and an axiomatic point of view. Hu et al. [71, 73] for their part studied fuzzy relations based on kernel functions.

In the following, we introduce a general implicator-conjunctor-based (IC) fuzzy rough set model which encapsulates all of them.

7.2 The IC model

Given a fuzzy relation R on U to describe the indiscernibility relation between the objects of the universe, we introduce the following fuzzy rough set model:

Definition 7.2.1. Let (U,R) be a fuzzy relation approximation space, $\mathscr{I}$ an implicator and $\mathscr{C}$ a conjunctor. The pair of *fuzzy relation-based approximation operators* $(\underline{\text{apr}}_{R,\mathscr{I}}, \overline{\text{apr}}_{R,\mathscr{C}})$ is defined by, for $X \in \mathscr{F}(U)$ and $x \in U$,

$$(\underline{\text{apr}}_{R,\mathscr{I}}(X))(x) = \inf_{y\in U} \mathscr{I}(R(y,x),X(y)), \tag{7.1}$$

$$(\overline{\text{apr}}_{R,\mathscr{C}}(X))(x) = \sup_{y\in U} \mathscr{C}(R(y,x),X(y)). \tag{7.2}$$

In the remainder of this work, we will refer to this model as the implicator-conjunctor-based (fuzzy rough set) model or shortly, IC model.

When the fuzzy set X we want to approximate is crisp and the relation R is a crisp equivalence relation E, this model coincides with Pawlak's rough set model: let $x \in U$,

$$\begin{aligned}
(\underline{\text{apr}}_{E,\mathscr{I}}(X))(x) = 1 &\Leftrightarrow \forall y \in U\colon \mathscr{I}(E(y,x),X(y)) = 1 \\
&\Leftrightarrow \forall y \in U\colon E(y,x) = 1 \Rightarrow X(y) = 1, \\
&\Leftrightarrow [x]_E \subseteq X, \\
(\overline{\text{apr}}_{E,\mathscr{C}}(X))(x) = 1 &\Leftrightarrow \exists y \in U\colon \mathscr{C}(E(y,x),X(y)) = 1 \\
&\Leftrightarrow \exists y \in U\colon E(y,x) = 1 \wedge X(y) = 1, \\
&\Leftrightarrow [x]_E \cap X \neq \emptyset.
\end{aligned}$$

Definition 7.2.1 covers many fuzzy rough set models which have been proposed in literature, and which emerge by choosing a specific implicator $\mathscr{I}$, conjunctor $\mathscr{C}$ and binary fuzzy relation R. We list these models in Table 7.1. For example, Dubois and Prade [41] used the Kleene-Dienes implicator and the minimum operator to replace the Boolean implication and conjunction respectively.

As can be seen from Table 7.1, Greco et al. [51, 52] were the first to consider reflexive binary fuzzy relations and Wu et al. [170, 171] were the first to consider general binary fuzzy relations. Also note that Greco et al. used arbitrary t-norms and t-conorms as aggregation operators instead of the infimum and supremum operators. Mi and Zhang [113] initiated the use of conjunctors which are not necessarily t-norms.

Remark 7.2.2. Note that some authors [73, 118, 192] require $\mathcal{T}$ to be lower semi-continuous instead of left-continuous in each parameter, i.e.,

$$(\forall a, b \in [0,1])(\forall \epsilon > 0)(\exists \delta > 0)(\forall c \in [0,1]): \\ a - \delta < c < a + \delta \Rightarrow \mathcal{T}(a,b) - \epsilon \leq \mathcal{T}(c,b).$$

By a result from [48] these two notions are equivalent for t-norms. Also, some papers [100, 112, 113, 169–171] consider fuzzy relations in $U \times W$, with both U and W non-empty, finite universes, but in this work we restrict ourselves to the case where $U = W$.

A variant of the IC model was proposed in [74] by Inuiguchi: the lower approximation $\underline{\text{apr}}^{\text{I}}_{R^*,\mathcal{I}}(X)$ of X by a fuzzy relation R^* in U is given by, for $x \in U$,

$$(\underline{\text{apr}}^{\text{I}}_{R^*,\mathcal{I}}(X))(x) = \min(X(x), \inf_{y \in U} \mathcal{I}(R^*(y,x), X(y))),$$

while the upper approximation $\overline{\text{apr}}^{\text{I}}_{R^*,\mathcal{C}}(X)$ of X by R^* is given by, for $x \in U$,

$$(\overline{\text{apr}}^{\text{I}}_{R^*,\mathcal{C}}(X))(x) = \max(X(x), \sup_{y \in U} \mathcal{C}(R^*(x,y), X(y))).$$

Note that this model can be seen as a special case of the IC model if in Definition 7.2.1 a border implicator $\mathcal{I}$ and a border conjunctor $\mathcal{C}$ are chosen and if the relation R defined by

$$\forall x, y \in U: R(x,y) = \max(R^*(x,y), \text{id}(x,y)),$$

is symmetric, where id is the fuzzy identity relation defined by

$$\forall x, y \in U: \ \text{id}(x,y) = \begin{cases} 1 & x = y, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, note that we can also consider the IC model with a fuzzy neighborhood operator instead of a fuzzy relation.

Table 7.1: Overview of implicator-conjunctor-based fuzzy rough set models in literature

Reference	$\mathscr{C}$	$\mathscr{I}$	R
[41, 42] Dubois & Prade, '90	$\mathscr{T}_M$	$\mathscr{I}_{KD}$	$\mathscr{T}_M$-sim.
[118] Morsi & Yakout, '98	left-cont. $\mathscr{T}$	R-impl.	$\mathscr{T}$-sim.
[51, 52] Greco et al., '98	$\mathscr{T}$	S-impl.	reflexive
[7] Boixader et al., '00	cont. $\mathscr{T}$	R-impl.	$\mathscr{T}$-sim.
[137] Radzikowska & Kerre, '02	$\mathscr{T}$	border	$\mathscr{T}_M$-sim.
[170, 171] Wu et al., '03	$\mathscr{T}_M$	S-impl.	general
[113] Mi & Zhang, '04	$\mathscr{C}$	R-impl.	general
[131] Pei, '05	$\mathscr{T}_M$	S-impl.	general
[169] Wu et al., '05	cont. $\mathscr{T}$	implicator	general
[192] Yeung et al., '05	left-cont. $\mathscr{T}$	S-impl.	general
[192] Yeung et al., '05	$\mathscr{C}$	R-impl.	general
[100] Liu, '06	$\mathscr{T}$	R-impl.	general
[24] De Cock et al., '07	$\mathscr{T}$	border	general
[98] Liu, '08	$\mathscr{T}_M$	S-impl.	general
[112] Mi et al., '08	cont. $\mathscr{T}$	S-impl.	general
[71, 73] Hu et al., '10	left-cont. $\mathscr{T}$	S-impl.	$\mathscr{T}_{\cos}$-sim.
[71, 73] Hu et al., '10	$\mathscr{C}$	R-impl.	$\mathscr{T}_{\cos}$-sim.

Definition 7.2.3. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ an implicator and $\mathscr{C}$ a conjunctor, then the pair of *fuzzy neighborhood-based approximation operators* $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ is defined by, for $X \in \mathscr{F}(U)$ and $x \in U$,

$$(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X))(x) = \inf_{y\in U} \mathscr{I}(\text{N}(x)(y), X(y)), \quad (7.3)$$

$$(\overline{\text{apr}}_{\text{N},\mathscr{C}}(X))(x) = \sup_{y\in U} \mathscr{C}(\text{N}(x)(y), X(y)). \quad (7.4)$$

All fuzzy neighborhood operators discussed in Chapter 6 can be used to define fuzzy neighborhood-based approximation operators. For the remainder of this work, we will work with the IC model using fuzzy neighborhood operators. Note that the models determined in Definitions 7.2.1 and 7.2.3 are interchangeable via $R(y,x) = \text{N}(x)(y)$ for $x, y \in U$. Next, we study the properties of the IC model.

7.3 Properties of the IC model

In this section, we consider the properties of the IC model defined with a fuzzy neighborhood operator. In particular, we discuss which properties of Pawlak's rough set model of Table 2.1 are maintained, and which conditions need to be imposed in order for the remaining ones to hold.

In Table 7.2, we list the adaptations of the properties in Table 2.1, with X and Y fuzzy sets in U and $(\underline{\text{apr}}, \overline{\text{apr}})$ a pair of fuzzy approximation operators on U. For the duality property, we assume $\mathscr{N}$ to be an involutive negator. For the property (IU), we consider the $\mathscr{T}_M$-intersection and the $\mathscr{S}_M$-union. The constant set property (CS) emerges by extending (UE) to every constant fuzzy set $\hat{\alpha}$ for $\alpha \in [0,1]$. If (CS) is satisfied, then a priori (UE) also holds.

The property (RM) of Table 2.1 can be extended in three ways: relation monotonicity (RM), neighborhood monotonicity (NM) and covering monotonicity (CM):

- Let R and R' be fuzzy relations on U, $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ a pair of approximation operators in the fuzzy relation approximation space (U,R) and $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ a pair of approximation operators in the fuzzy relation approximation space

(U,R'), then the property (RM) is given by

$$R \subseteq R' \Rightarrow \begin{cases} \forall X \in \mathscr{F}(U)\colon \underline{\text{apr}}_2(X) \subseteq \underline{\text{apr}}_1(X) \\ \forall X \in \mathscr{F}(U)\colon \overline{\text{apr}}_1(X) \subseteq \overline{\text{apr}}_2(X) \end{cases} \tag{7.5}$$

- Let N and N′ be two fuzzy neighborhood operators on U, $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ a pair of fuzzy approximation operators based on the fuzzy neighborhood operator N and $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ a pair of fuzzy approximation operators based on the fuzzy neighborhood operator N′, then the property (NM) is given by

$$\text{N} \preceq \text{N}' \Rightarrow \begin{cases} \forall X \in \mathscr{F}(U)\colon \underline{\text{apr}}_2(X) \subseteq \underline{\text{apr}}_1(X) \\ \forall X \in \mathscr{F}(U)\colon \overline{\text{apr}}_1(X) \subseteq \overline{\text{apr}}_2(X) \end{cases} \tag{7.6}$$

- Let $\mathbb{C}$ and $\mathbb{C}'$ be two coverings of U, $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ a pair of fuzzy approximation operators in the fuzzy covering approximation space $(U,\mathbb{C})$ and $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ a pair of fuzzy approximation operators in the fuzzy covering approximation space $(U,\mathbb{C}')$, then the property (CM) is given by

$$\mathbb{C} \sqsubseteq \mathbb{C}' \Rightarrow \begin{cases} \forall X \in \mathscr{F}(U)\colon \underline{\text{apr}}_2(X) \subseteq \underline{\text{apr}}_1(X) \\ \forall X \in \mathscr{F}(U)\colon \overline{\text{apr}}_1(X) \subseteq \overline{\text{apr}}_2(X) \end{cases} \tag{7.7}$$

  As in the crisp setting, we write $\mathbb{C} \sqsubseteq \mathbb{C}'$ if and only if

$$(\forall K \in \mathbb{C})(\exists K' \in \mathbb{C}')(K \subseteq K'). \tag{7.8}$$

All three monotonicity properties have a similar interpretation: the more objects are discernible from each other, the more accurate the approximations. Moreover, all are interchangeable, since we can define a fuzzy relation based on a fuzzy neighborhood operator and vice versa, and we can define a covering $\{\text{N}(x) \mid x \in U\}$ for a reflexive neighborhood operator N. Depending on the considered theoretical framework, we will choose one of the three. For instance, in this chapter we will consider the property (NM).

Most of these properties have been studied for the IC model in one form or another, however, only sufficient conditions on $\mathscr{I}$, $\mathscr{C}$ and N were provided. We

Table 7.2: Properties of the fuzzy approximation operators $(\underline{\text{apr}}, \overline{\text{apr}})$ on U

Property		Definition
Duality	(D)	$\overline{\text{apr}}(X) = (\underline{\text{apr}}(X^{\mathcal{N}}))^{\mathcal{N}}$
Inclusion	(INC)	$\underline{\text{apr}}(X) \subseteq X$ and $X \subseteq \overline{\text{apr}}(X)$
Set monotonicity	(SM)	$X \subseteq Y \Rightarrow \begin{cases} \underline{\text{apr}}(X) \subseteq \underline{\text{apr}}(Y) \\ \overline{\text{apr}}(X) \subseteq \overline{\text{apr}}(Y) \end{cases}$
Intersection and union	(IU)	$\underline{\text{apr}}(X \cap Y) = \underline{\text{apr}}(X) \cap \underline{\text{apr}}(Y)$ $\overline{\text{apr}}(X \cup Y) = \overline{\text{apr}}(X) \cup \overline{\text{apr}}(Y)$
Idempotence	(ID)	$\underline{\text{apr}}(\underline{\text{apr}}(X)) \supseteq \underline{\text{apr}}(X)$ $\overline{\text{apr}}(\overline{\text{apr}}(X)) \subseteq \overline{\text{apr}}(X)$
Interaction lower and upper	(LU)	$\overline{\text{apr}}(\underline{\text{apr}}(X)) \subseteq \underline{\text{apr}}(X)$ $\underline{\text{apr}}(\overline{\text{apr}}(X)) \supseteq \overline{\text{apr}}(X)$
Constant set	(CS)	$\underline{\text{apr}}(\hat{\alpha}) = \hat{\alpha}$ and $\overline{\text{apr}}(\hat{\alpha}) = \hat{\alpha}$
Universe and empty set	(UE)	$\underline{\text{apr}}(U) = U$ and $\overline{\text{apr}}(U) = U$ $\underline{\text{apr}}(\emptyset) = \emptyset$ and $\overline{\text{apr}}(\emptyset) = \emptyset$
Adjointness	(A)	$\overline{\text{apr}}(X) \subseteq Y \Leftrightarrow X \subseteq \underline{\text{apr}}(Y)$

prove for the properties of $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ which conditions on the fuzzy neighborhood operator N are necessary and sufficient, given conditions on the fuzzy logical connectives $\mathscr{I}$ and $\mathscr{C}$. This way, the results stated in Proposition 4.7.1 are extended to the fuzzy setting.

First, we discuss the duality property.

Proposition 7.3.1. Let N be a fuzzy neighborhood operator on U and $\mathscr{N}$ an involutive negator. Let $\mathscr{I}$ be an implicator and $\mathscr{C}$ the induced conjunctor of $\mathscr{I}$ and $\mathscr{N}$, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (D).

Proof. Let $X \in \mathscr{F}(U)$ and $x \in U$, then

$$\begin{aligned}
(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X^{\mathscr{N}}))^{\mathscr{N}}(x) &= \mathscr{N}\left(\inf_{y\in U}\mathscr{I}(\text{N}(x)(y), \mathscr{N}(X(y)))\right) \\
&= \sup_{y\in U}\mathscr{N}(\mathscr{I}(\text{N}(x)(y), \mathscr{N}(X(y)))) \\
&= \sup_{y\in U}\mathscr{C}(\text{N}(x)(y), X(y)) \\
&= (\overline{\text{apr}}_{\text{N},\mathscr{C}}(X))(x).
\end{aligned}$$

□

Proposition 7.3.1 holds when we consider the following fuzzy logical connectives:

- $\mathscr{C}$ a IMTL-t-norm, $\mathscr{I}$ its R-implicator and $\mathscr{N}$ the induced negator of $\mathscr{I}$,
- $\mathscr{N}$ an involutive negator, $\mathscr{C}$ a t-norm $\mathscr{T}$ and $\mathscr{I}$ the S-implicator based on the $\mathscr{N}$-dual t-conorm of $\mathscr{T}$ and the negator $\mathscr{N}$.

Next, the inclusion property is discussed.

Proposition 7.3.2. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ a border implicator and $\mathscr{C}$ a border conjunctor, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (INC) if and only if N is reflexive.

Proof. Assume that the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (INC), then for $x \in U$, it holds that

$$\begin{aligned} 1 = 1_x(x) &\leq (\overline{\text{apr}}_{\text{N},\mathscr{C}}(1_x))(x) \\ &= \sup_{y\in U} \mathscr{C}(\text{N}(x)(y), 1_x(y)) \\ &= \mathscr{C}(\text{N}(x)(x), 1) \\ &= \text{N}(x)(x), \end{aligned}$$

hence, we conclude that N is reflexive.

On the other hand, assume N is reflexive. Let $X \in \mathscr{F}(U)$ and $x \in U$, then

$$(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X))(x) \leq \mathscr{I}(\text{N}(x)(x), X(x)) = \mathscr{I}(1, X(x)) = X(x),$$
$$(\overline{\text{apr}}_{\text{N},\mathscr{C}}(X))(x) \geq \mathscr{C}(\text{N}(x)(x), X(x)) = \mathscr{C}(1, X(x)) = X(x).$$

We conclude that the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (INC). □

In the following, it is verified that the properties (SM) and (IU) always hold.

Proposition 7.3.3. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ an implicator and $\mathscr{C}$ a conjunctor, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (SM).

Proof. Every implicator $\mathscr{I}$ and every conjunctor $\mathscr{C}$ is increasing in the second parameter. □

Proposition 7.3.4. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ an implicator and $\mathscr{C}$ a conjunctor, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (IU).

Proof. Since the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{C}})$ satisfies (SM), it holds for all fuzzy sets $X, Y \in \mathscr{F}(U)$ that

(a) $\underline{\text{apr}}_{\text{N},\mathscr{I}}(X \cap Y) \subseteq \underline{\text{apr}}_{\text{N},\mathscr{I}}(X) \cap \underline{\text{apr}}_{\text{N},\mathscr{I}}(Y)$,

(b) $\overline{\text{apr}}_{\text{N},\mathscr{C}}(X \cup Y) \supseteq \overline{\text{apr}}_{\text{N},\mathscr{C}}(X) \cup \overline{\text{apr}}_{\text{N},\mathscr{C}}(Y)$.

On the other hand, let $x \in U$, then

$$(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X \cap Y))(x) = \inf_{y\in U} \mathscr{I}(\text{N}(x)(y), \min(X(y), Y(y)))$$

$$\begin{aligned}
&= \min\Big(\inf_{y\in U,\ X(y)\le Y(y)} \mathscr{I}(\mathrm{N}(x)(y), X(y)),\\
&\qquad \inf_{y\in U,\ Y(y)\le X(y)} \mathscr{I}(\mathrm{N}(x)(y), Y(y))\Big)\\
&\geq \min\Big(\inf_{y\in U} \mathscr{I}(\mathrm{N}(x)(y), X(y)),\\
&\qquad \inf_{y\in U} \mathscr{I}(\mathrm{N}(x)(y), Y(y))\Big)\\
&= \min\Big((\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X))(x), (\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(Y))(x)\Big)\\
&= (\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X) \cap \underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(Y))(x).
\end{aligned}$$

Similarly, we obtain $(\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{C}}(X\cup Y))(x) \leq (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{C}}(X) \cup \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{C}}(Y))(x)$. □

Next, we discuss the idempotence property.

Proposition 7.3.5. Let N be a fuzzy neighborhood operator on U and $\mathscr{C}$ a left-continuous t-norm $\mathscr{T}$.

(a) If $\mathscr{I}$ is the R-implicator of $\mathscr{T}$, then the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$ satisfies (ID) if and only if N is $\mathscr{T}$-transitive.

(b) If $\mathscr{N}$ is an involutive negator and $\mathscr{I}$ is the S-implicator based on the $\mathscr{N}$-dual of $\mathscr{T}$ and $\mathscr{N}$, then the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$ satisfies (ID) if and only if N is $\mathscr{T}$-transitive.

Proof. Let $\mathscr{T}$ be a left-continuous t-norm and $\mathscr{I}$ its R-implicator. First, assume that the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$ satisfies (ID). Let $x, y \in U$, then it holds that

$$(\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(1_y)))(x) \leq (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(1_y))(x),$$

i.e.,

$$\sup_{z\in U} \mathscr{T}\Big(\mathrm{N}(x)(z), \sup_{v\in U} \mathscr{T}(\mathrm{N}(z)(v), 1_y(v))\Big) \leq \sup_{u\in U} \mathscr{T}(\mathrm{N}(x)(u), 1_y(u)).$$

By the definition of 1_y, we obtain that

$$\sup_{z\in U} \mathscr{T}(\mathrm{N}(x)(z), \mathrm{N}(z)(y)) \leq \mathrm{N}(x)(y).$$

We conclude that N is $\mathscr{T}$-transitive.

On the other hand, assume N is $\mathscr{T}$-transitive. For $X \in \mathscr{F}(U)$ and $x \in U$, it holds that

$$\begin{aligned}
(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X))(x) &= \inf_{y\in U} \mathscr{I}(\text{N}(x)(y), X(y)) \\
&\leq \inf_{y\in U} \mathscr{I}\Big(\sup_{z\in U} \mathscr{T}(\text{N}(x)(z), \text{N}(z)(y)), X(y)\Big) \\
&= \inf_{y\in U} \inf_{z\in U} \mathscr{I}(\mathscr{T}(\text{N}(x)(z), \text{N}(z)(y)), X(y)) \\
&= \inf_{z\in U} \inf_{y\in U} \mathscr{I}(\text{N}(x)(z), \mathscr{I}(\text{N}(z)(y), X(y))) \\
&= \inf_{z\in U} \mathscr{I}(\text{N}(x)(z), \inf_{y\in U} \mathscr{I}(\text{N}(z)(y), X(y))) \\
&= \inf_{z\in U} \mathscr{I}(\text{N}(x)(z), \underline{\text{apr}}_{\text{N},\mathscr{I}}(X)(z)) \\
&= (\underline{\text{apr}}_{\text{N},\mathscr{I}}(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X)))(x).
\end{aligned}$$

In a similar way, we can prove that

$$(\overline{\text{apr}}_{\text{N},\mathscr{T}}(\overline{\text{apr}}_{\text{N},\mathscr{T}}(X)))(x) \leq (\overline{\text{apr}}_{\text{N},\mathscr{T}}(X))(x).$$

The second statement can be proved analogously. □

In the following, we discuss the property (LU).

Proposition 7.3.6. Let N be a fuzzy neighborhood operator on U, $\mathscr{C}$ a left-continuous t-norm $\mathscr{T}$ and $\mathscr{I}$ its R-implicator. If N is symmetric, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ satisfies (LU) if and only if N is $\mathscr{T}$-transitive.

Proof. Assume that N is symmetric and the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ satisfies (LU). Let $x, y \in U$, then

$$\begin{aligned}
\text{N}(x)(y) &= (\overline{\text{apr}}_{\text{N},\mathscr{T}}(1_y))(x) \\
&\leq \inf_{z\in U} \mathscr{I}(\text{N}(x)(z), (\overline{\text{apr}}_{\text{N},\mathscr{T}}(1_y))(z)) \\
&= \inf_{z\in U} \mathscr{I}(\text{N}(x)(z), \text{N}(z)(y)).
\end{aligned}$$

Hence, by the residuation principle we obtain that for all $z \in U$:

$$\mathscr{T}(\text{N}(x)(y), \text{N}(x)(z)) \leq \text{N}(z)(y).$$

Since $\mathscr{T}$ is commutative and N is symmetric, it holds for $z \in U$ that

$$\mathscr{T}(\mathrm{N}(z)(x), \mathrm{N}(x)(y)) \leq \mathrm{N}(z)(y).$$

Since x, y and z are chosen arbitrarily, we conclude that N is $\mathscr{T}$-transitive.

On the other hand, assume N is symmetric and $\mathscr{T}$-transitive. Note that for a left-continuous t-norm $\mathscr{T}$ and its R-implicator $\mathscr{I}$ it holds that (see [137, 138])

$$\forall a, b, c \in [0,1]\colon \mathscr{T}(a, \mathscr{I}(b,c)) \leq \mathscr{I}(\mathscr{I}(a,b),c).$$

Moreover, we have an alternative characterisation for $\mathscr{T}$-transitivity:

$$\forall x, y \in U\colon \inf_{z \in U} \mathscr{I}(\mathrm{N}(x)(z), \mathrm{N}(z)(y)) = \mathrm{N}(x)(y).$$

Let $X \in \mathscr{F}(U)$ and $x \in U$, then

$$\begin{aligned}
(\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X)))(x) &= \sup_{y \in U} \mathscr{T}\Big(\mathrm{N}(x)(y), \inf_{z \in U} \mathscr{I}(\mathrm{N}(y)(z), X(z))\Big) \\
&\leq \sup_{y \in U} \inf_{z \in U} \mathscr{T}\Big(\mathrm{N}(x)(y), \mathscr{I}(\mathrm{N}(y)(z), X(z))\Big) \\
&\leq \sup_{y \in U} \inf_{z \in U} \mathscr{I}\Big(\mathscr{I}(\mathrm{N}(x)(y), \mathrm{N}(y)(z)), X(z)\Big) \\
&\leq \inf_{z \in U} \mathscr{I}\Big(\inf_{y \in U} \mathscr{I}(\mathrm{N}(x)(y), \mathrm{N}(y)(z)), X(z)\Big) \\
&= \inf_{z \in U} \mathscr{I}(\mathrm{N}(x)(z), X(z)) \\
&= (\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X))(x).
\end{aligned}$$

For the other inclusion, let $y \in U$, then

$$\begin{aligned}
\mathscr{T}(\mathrm{N}(x)(y), (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(x)) &= \mathscr{T}\Big(\mathrm{N}(x)(y), \sup_{z \in U} \mathscr{T}(\mathrm{N}(x)(z), X(z))\Big) \\
&= \sup_{z \in U} \mathscr{T}\Big(\mathrm{N}(x)(y), \mathscr{T}(\mathrm{N}(x)(z), X(z))\Big) \\
&= \sup_{z \in U} \mathscr{T}(\mathscr{T}(\mathrm{N}(x)(y), \mathrm{N}(x)(z)), X(z)) \\
&= \sup_{z \in U} \mathscr{T}(\mathscr{T}(\mathrm{N}(y)(x), \mathrm{N}(x)(z)), X(z)) \\
&\leq \sup_{z \in U} \mathscr{T}(\mathrm{N}(y)(z), X(z)) \\
&= (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(y).
\end{aligned}$$

By this result and the definition of an R-implicator, it holds that

$$\begin{aligned} & \mathscr{I}(\mathrm{N}(x)(y), (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(y)) \\ = \ & \sup\{d \in [0,1] \mid \mathscr{T}(\mathrm{N}(x)(y), d) \leq (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(y)\} \\ \geq \ & (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(x). \end{aligned}$$

Hence, since y was chosen arbitrarily:

$$\begin{aligned} (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(x) &\leq \inf_{y \in U} \mathscr{I}(\mathrm{N}(x)(y), (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X))(y)) \\ &= (\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(X)))(x). \end{aligned}$$

We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$ satisfies (LU). □

The previous proposition does not hold for a left-continuous t-norm and an S-implicator, as illustrated in the next example:

Example 7.3.7. Let $U = \{x, y\}$, $\mathscr{T}$ the minimum operator, $\mathscr{I}$ the Kleene-Dienes implicator and N a fuzzy neighborhood operator on U such that

- $\mathrm{N}(x)(x) = \mathrm{N}(y)(y) = 1$,
- $\mathrm{N}(x)(y) = \mathrm{N}(y)(x) = 0.5$,

then N is symmetric and $\mathscr{T}$-transitive. For the fuzzy set $X = 0.2/x + 0.6/y$, we obtain that $\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X) = 0.2/x + 0.5/y$ and $\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X)) = 0.5/x + 0.5/y$. Hence, the inclusion

$$\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X)) \subseteq \underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}(X)$$

does not hold. We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$ does not satisfy (LU) for this choice of t-norm and implicator.

Next, we study the properties (CS) and (UE).

Proposition 7.3.8. Let N be a fuzzy neighborhood operator on U, $\mathscr{I}$ a border implicator and $\mathscr{C}$ a border conjunctor. If N is normalized, then the pair $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{C}})$ satisfies (CS) and (UE).

Proof. Let $\alpha \in [0,1]$ and $x \in U$. As N is normalized, the fuzzy set $N(x)$ is normalized, hence, there exists $y \in U$ such that $N(x)(y) = 1$. We obtain that

$$(\underline{\text{apr}}_{N,\mathscr{I}}(\hat{\alpha}))(x) \leq \mathscr{I}(N(x)(y),\alpha) = \mathscr{I}(1,\alpha) = \alpha,$$
$$(\overline{\text{apr}}_{N,\mathscr{C}}(\hat{\alpha}))(x) \geq \mathscr{C}(N(x)(y),\alpha) = \mathscr{C}(1,\alpha) = \alpha.$$

On the other hand, as for all $z \in U$ it holds that $N(x)(z) \leq 1$, we obtain that $\underline{\text{apr}}_{N,\mathscr{I}}(\hat{\alpha}) \supseteq \hat{\alpha}$, since an implicator is decreasing in the first parameter, and $\overline{\text{apr}}_{N,\mathscr{C}}(\hat{\alpha}) \subseteq \hat{\alpha}$, since a conjunctor is increasing in the first parameter. □

Note that the inverse statement of Proposition 7.3.8 does not hold.

Example 7.3.9. Let $U = \{x_n \mid n \in \mathbb{N} \setminus \{0\}\}$. For each $x_n, x_m \in U$, let

$$N(x_n)(x_m) = 1 - \frac{1}{m},$$

then N is not normalized. Let $\mathscr{C}$ be the minimum operator and $\mathscr{I}$ the Gödel implicator. We illustrate that the pair $(\underline{\text{apr}}_{N,\mathscr{I}}, \overline{\text{apr}}_{N,\mathscr{T}})$ satisfies (CS). Let $\alpha \in [0,1]$ and $x_n \in U$,

$$\begin{aligned}
(\overline{\text{apr}}_{N,\mathscr{T}}(\hat{\alpha}))(x_n) &= \sup_{x_m \in U} \min(N(x_n)(x_m), \alpha) \\
&= \min\left(\sup_{x_m \in U} N(x_n)(x_m), \alpha\right) \\
&= \min\left(\sup_{m \in \mathbb{N}\setminus\{0\}} (1 - \frac{1}{m}), \alpha\right) \\
&= \min(1, \alpha) \\
&= \alpha.
\end{aligned}$$

Moreover, note that $\underline{\text{apr}}_{N,\mathscr{I}}(\hat{\alpha}) \supseteq \hat{\alpha}$, as $\mathscr{I}$ is decreasing in the first parameter. For $\alpha = 1$, it holds that $\underline{\text{apr}}_{N,\mathscr{I}}(\hat{\alpha}) = \hat{\alpha}$, since $\mathscr{I}(a,1) = 1$ for all $a \in [0,1]$. Let $\alpha < 1$ and let $m \in \mathbb{N} \setminus \{0\}$ such that $\alpha < 1 - \frac{1}{m}$, then for $x_n \in U$,

$$(\underline{\text{apr}}_{N,\mathscr{I}}(\hat{\alpha}))(x_n) \leq \mathscr{I}(N(x_n)(x_m), \alpha) = \alpha.$$

Hence, we conclude that the pair $(\underline{\text{apr}}_{N,\mathscr{I}}, \overline{\text{apr}}_{N,\mathscr{T}})$ satisfies (CS), although the fuzzy neighborhood operator N is not normalized.

We study the adjointness property next.

Proposition 7.3.10. Let N be a fuzzy neighborhood operator on U, $\mathscr{C}$ a left-continuous t-norm $\mathscr{T}$ and $\mathscr{I}$ its R-implicator, then the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ satisfies (A) if and only if N is symmetric.

Proof. Assume that the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ satisfies (A). Let $x, y \in U$, then for $X = 1_x$ and $Y = \text{N}(x)$ it holds that

$$\forall z \in U\colon (\overline{\text{apr}}_{\text{N},\mathscr{T}}(X))(z) \leq Y(z) \Leftrightarrow \forall z \in U\colon X(z) \leq (\underline{\text{apr}}_{\text{N},\mathscr{I}}(Y))(z),$$

i.e.,

$$\begin{aligned} & \forall z \in U\colon \sup_{v\in U} \mathscr{T}(\text{N}(z)(v), 1_x(v)) \leq \text{N}(x)(z) \\ \Leftrightarrow \quad & \forall z \in U\colon 1_x(z) \leq \inf_{v\in U} \mathscr{I}(\text{N}(z)(v), \text{N}(x)(v)). \end{aligned}$$

By the definition of 1_x, we obtain that

$$\forall z \in U\colon \text{N}(x)(z) \leq \text{N}(z)(x) \Leftrightarrow 1 \leq \inf_{v\in U} \mathscr{I}(\text{N}(x)(v), \text{N}(x)(v)).$$

Since $\mathscr{I}$ is an R-implicator, the left-hand-side of this equivalence is true, hence, for $y \in U$ we obtain that $\text{N}(x)(y) \leq \text{N}(y)(x)$. By changing the roles of the objects x and y, we obtain in a similar way that $\text{N}(y)(x) \leq \text{N}(x)(y)$. We conclude that N is symmetric.

On the other hand, assume N is symmetric. For $X, Y \in \mathscr{F}(U)$ it holds that

$$\begin{aligned} \overline{\text{apr}}_{\text{N},\mathscr{T}}(X) \subseteq Y \quad & \Leftrightarrow \quad \forall x, y \in U\colon \mathscr{T}(\text{N}(x)(y), X(y)) \leq Y(x) \\ & \Leftrightarrow \quad \forall x, y \in U\colon \mathscr{T}(X(y), \text{N}(y)(x)) \leq Y(x) \\ & \Leftrightarrow \quad \forall x, y \in U\colon X(y) \leq \mathscr{I}(\text{N}(y)(x), Y(x)) \\ & \Leftrightarrow \quad X \subseteq \underline{\text{apr}}_{\text{N},\mathscr{I}}(Y). \end{aligned}$$

We conclude that the pair $(\underline{\text{apr}}_{\text{N},\mathscr{I}}, \overline{\text{apr}}_{\text{N},\mathscr{T}})$ satisfies (A). □

Note that in the proof of Proposition 7.3.10 we explicitly used the residuation principle. In general, this proposition does not hold for other combinations of $\mathscr{T}$ and $\mathscr{I}$.

Example 7.3.11. Let $U = \{x, y\}$ and N a fuzzy neighborhood operator on U such that $N(x)(x) = N(y)(y) = 1$ and $N(x)(y) = N(y)(x) = 0.5$, then N is symmetric. Let $\mathcal{T}$ be the minimum operator and $\mathcal{I}$ the Kleene-Dienes implicator. For the fuzzy set $X = 0.3/x + 0.4/y$, we obtain that $\underline{\text{apr}}_{N,\mathcal{I}}(X) = 0.3/x + 0.4/y$ and $\overline{\text{apr}}_{N,\mathcal{T}}(X) = 0.4/x + 0.4/y$. Hence, $X \subseteq \underline{\text{apr}}_{N,\mathcal{I}}(X)$, but $\overline{\text{apr}}_{N,\mathcal{T}}(X) \nsubseteq X$. We conclude that the pair $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{T}})$ does not satisfy (A) for this choice of t-norm and implicator.

Finally, we discuss the property (NM).

Proposition 7.3.12. Let N be a fuzzy neighborhood operator on U, $\mathcal{I}$ an implicator and $\mathcal{C}$ a conjunctor, then the pair $(\underline{\text{apr}}_{N,\mathcal{I}}, \overline{\text{apr}}_{N,\mathcal{C}})$ satisfies (NM).

Proof. Every implicator $\mathcal{I}$ is decreasing in the first parameter and every conjunctor $\mathcal{C}$ is decreasing in the second parameter. □

7.4 Conclusions

In this chapter, we have presented a historical overview of fuzzy rough set theory since the late 1980s. We have introduced an implicator-conjunctor-based fuzzy rough set model which encapsulates many fuzzy rough set models described in literature. Moreover, we have studied the properties of this model when a fuzzy neighborhood operator is considered. We conclude that all properties in Table 7.2 are satisfied when $\mathcal{C}$ is an IMTL-t-norm $\mathcal{T}$, $\mathcal{I}$ is its R-implicator, $\mathcal{N}$ is its induced negator and N is reflexive, symmetric and $\mathcal{T}$-transitive. If we omit the duality property, it is sufficient that $\mathcal{T}$ is left-continuous instead of begin an IMTL-t-norm.

CHAPTER 8

Fuzzy covering-based rough set models

In literature, different models are defined in which a fuzzy covering is used to describe the indiscernibility between objects. Besides using fuzzy neighborhood operators based on a fuzzy covering to define the implicator-conjunctor-based fuzzy rough set model, we can extend the tight and loose granule-based approximation operators for a fuzzy covering. Fuzzy extensions of the tight covering-based approximation operators have been studied by Li et al. [91], Inuiguchi et al. [74, 75] and Wu et al. [168] which we recall here. Moreover, we introduce two tight fuzzy covering-based rough set models: one using the theory of representation by levels introduced by Sánchez et al. [144] and one constructed from an intuitive point of view. In addition, we recall the loose fuzzy covering-based rough set model of Li et al. [91] and introduce a new loose fuzzy covering-based rough set model using representation by levels.

For each fuzzy covering-based rough set model we study in Sections 8.1 and 8.2 which properties of Pawlak's rough set model are maintained. From Table 4.5 we obtain that the pair of tight granule-based approximation operators satisfies the properties (D), (INC), (SM), (ID) and (UE) for a crisp covering. Hence, given a fuzzy covering, we study whether the fuzzy extensions of these approximation operators satisfy the properties (D), (INC), (SM), (ID), (CS) and (UE). Moreover, the pair of loose granule-based approximation operators satisfies the properties (D), (INC), (SM), (IU), (UE), (A) and (CM). Thus, for the fuzzy extensions of these approximation operators we will study the properties (D), (INC), (SM), (IU), (CS), (UE), (A) and (CM) for a fuzzy covering. To this aim, we do not consider the property (NM) given in Eq. (7.6), but instead we study the property of covering monotonicity.

In addition, in Chapter 4 it is shown that the pair of loose granule-based approximation operators coincides with a pair of element-based approximation operators. We will show that this result is maintained in the fuzzy setting.

Furthermore, in Section 8.3 we will discuss partial order relations with respect to $\leq$ between fuzzy covering-based approximation operators. Let apr_1 and apr_2 be fuzzy approximation operators, then we write $\mathrm{apr}_1 \leq \mathrm{apr}_2$ if and only if $\forall X \subseteq U\colon \mathrm{apr}_1(X) \subseteq \mathrm{apr}_2(X)$. Given two pairs of dual fuzzy approximation operators $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1)$ and $(\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$, then we will write

$$\begin{aligned}(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1) \leq (\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2) \quad &\Leftrightarrow \quad \forall X \in \mathscr{F}(U)\colon \underline{\mathrm{apr}}_1(X) \subseteq \underline{\mathrm{apr}}_2(X) \\ &\Leftrightarrow \quad \forall X \in \mathscr{F}(U)\colon \overline{\mathrm{apr}}_2(X) \subseteq \overline{\mathrm{apr}}_1(X).\end{aligned}$$

When $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1) \leq (\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$, we say that the pair $(\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$ yields more accurate approximations than the pair $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1)$.

We will discuss the comparability of the IC model defined with the 17 fuzzy neighborhood operators discussed in Chapter 6 to the fuzzy covering-based approximation operators studied in Sections 8.1 and 8.2.

To end, we will state conclusions and future work objectives in Section 8.4.

8.1 Fuzzy extensions of tight granule-based approximation operators

In this section, we study five fuzzy covering-based rough set models which all extend the tight granule-based approximation operators defined in Eqs. (2.19) and (2.20) to the fuzzy setting.

8.1.1 Model of Li et al.

The first fuzzy covering-based rough set model we discuss was introduced by Li et al. [91].

Definition 8.1.1. [91] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathcal{T}$ a t-norm and $\mathcal{I}$ an implicator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}})$ is defined as follows: let $X \in \mathcal{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \inf_{y\in U} \mathcal{I}(K(y), X(y))),\\ (\overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) &= \inf_{K\in\mathbb{C}} \mathcal{I}(K(x), \sup_{y\in U} \mathcal{T}(K(y), X(y))).\end{aligned}$$

This model was proposed by the authors as a more general model than the models discussed in [18, 36], where a fuzzy covering related with a fuzzy relation was used. The properties of this model are given in the following proposition.

Proposition 8.1.2. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathcal{T}$ a t-norm and $\mathcal{I}$ an implicator.

- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}})$ satisfies (D) with respect to the involutive negator $\mathcal{N}$ if $\mathcal{T}$ is an IMTL-t-norm, $\mathcal{I}$ is its R-implicator and $\mathcal{N}$ equals the negator induced by $\mathcal{I}$.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}})$ satisfies (D) with respect to the involutive negator $\mathcal{N}$ if $\mathcal{I}$ is the S-implicator with respect to the t-conorm $\mathcal{S}$ and the negator $\mathcal{N}$, where $\mathcal{S}$ is the $\mathcal{N}$-dual of $\mathcal{T}$.

- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}})$ satisfies (INC) if $\mathscr{T}$ is left-continuous and $\mathscr{I}$ is its R-implicator.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}})$ satisfies (SM).
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}})$ satisfies (ID) if $\mathscr{T}$ is left-continuous and $\mathscr{I}$ is its R-implicator.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}})$ satisfies (CS) and (UE) if $\mathscr{T}$ is left-continuous and $\mathscr{I}$ is its R-implicator.

Proof. The properties (D), (INC), (SM) and (ID) were proven in [91]. We prove the last statement. Let $\mathscr{T}$ be a left-continuous t-norm, $\mathscr{I}$ its R-implicator and $\alpha \in [0,1]$. Since the inclusion property holds, it is sufficient to prove that $\hat{\alpha} \subseteq \underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\hat{\alpha})$ and $\overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\hat{\alpha}) \subseteq \hat{\alpha}$. Let $x \in U$ and let $K^* \in \mathbb{C}$ such that $K^*(x) = 1$, then

$$\begin{aligned}
(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\hat{\alpha}))(x) &\geq \mathscr{T}(K^*(x), \inf_{y\in U} \mathscr{I}(K^*(y), \alpha)) \\
&= \mathscr{T}(1, \inf_{y\in U} \mathscr{I}(K^*(y), \alpha)) \\
&= \inf_{y\in U} \mathscr{I}(K^*(y), \alpha) \\
&\geq \inf_{y\in U} \mathscr{I}(1, \alpha) \\
&= \alpha,
\end{aligned}$$

where we have used that $K^*(y) \leq 1$ for all $y \in U$. The proof for the inclusion $\overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\hat{\alpha}) \subseteq \hat{\alpha}$ is analogous. □

Note that for the properties (INC), (ID), (CS) and (UE) it is essential to consider a left-continuous t-norm and its R-implicator, as illustrated in the next example.

Example 8.1.3. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 1/x + 0.4/y$, $K_2 = 0.4/x + 1/y$ and $K_3 = 0.2/x + 0.7/y$. Let $\mathscr{T}$ be the minimum operator and $\mathscr{I}$ the Kleene-Dienes implicator, then for $X = U$ it holds that

$$\overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X) = 0.8/x + 0.7/y.$$

Hence, $X \nsubseteq \overline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X)$. Thus, (INC), (UE) and (CS) do not hold.

On the other hand, let $\mathbb{C} = \{K_1, K_2, K_3\}$ with $K_1 = 1/x + 1/y$, $K_2 = 0.4/x + 0.4/y$ and $K_3 = 0.2/x + 0.7/y$ and consider the product t-norm and the S-implicator $\mathscr{I}$ based on the probabilistic sum and the standard negator, i.e., $\mathscr{I}(a, b) = 1 - a + a \cdot b$ for $a, b \in [0, 1]$. For $X = 0.6/x + 1/y$ it holds that $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X) = 0.6/x + 0.644/y$ and $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X)) = 0.6/x + 0.6/y$. Hence, the inclusion

$$\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X) \subseteq \underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))$$

does not hold, thus, the property (ID) is not satisfied.

8.1.2 Model of Inuiguchi et al.

Next, we study the model of Inuiguchi et al. [74, 75]. They used the following logical connective: let $\mathscr{I}$ be an implicator, then the fuzzy logical connective $\xi[\mathscr{I}] \colon [0, 1] \times [0, 1] \to [0, 1]$ is defined by

$$\forall a, b \in [0, 1] \colon \xi[\mathscr{I}](a, b) = \inf\{c \in [0, 1] \mid \mathscr{I}(a, c) \geq b\}.$$

The connective $\xi[\mathscr{I}]$ is a conjunctor if $\forall a \in [0, 1) \colon \xi[\mathscr{I}](1, a) < 1$ (see [74]). We will assume that this always holds.

Furthermore, when $\mathscr{I}$ is upper semi-continuous, i.e., $\mathscr{I}$ is left-continuous in the first parameter and right-continuous[5] in the second, then the following equivalence holds [75]:

$$\forall a, b, c \in [0, 1] \colon \xi[\mathscr{I}](a, b) \leq c \Leftrightarrow \mathscr{I}(a, c) \geq b.$$

The model of Inuiguchi et al. is given in the following definition:

Definition 8.1.4. [74, 75] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{I}$ an upper semi-continuous implicator and $\mathscr{N}$ an involutive negator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}})$ is defined as follows: let $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}(X))(x) &= \sup_{K \in \mathbb{C}} \xi[\mathscr{I}](K(x), \inf_{y \in U} \mathscr{I}(K(y), X(y))),\\ (\overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}}(X))(x) &= (\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}(X^{\mathscr{N}}))^{\mathscr{N}}(x).\end{aligned}$$

[5] The definition of right-continuity is very similar to the definition of left-continuity presented in Eq. (5.1), only c is taken in the open interval $(a, a + \delta)$ instead of the interval $(a - \delta, a)$.

In [74] and [75], a collection $\mathscr{F} \subseteq \mathscr{F}(U)$ was used to define the operators. However, we will always assume that the collection $\mathscr{F}$ is a fuzzy covering. Next, we study the properties of this fuzzy covering-based rough set model.

Proposition 8.1.5. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{I}$ an upper semi-continuous implicator and $\mathscr{N}$ an involutive negator.

- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}})$ satisfies (D) with respect to $\mathscr{N}$.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}})$ satisfies (INC), (SM), (ID) and (UE).
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}})$ satisfies (CS) if $\mathscr{I}$ is a border implicator.

Proof. The properties (D), (INC), (SM), (ID) and (UE) were proven in [74, 75]. We prove the last statement. Let $\mathscr{I}$ be an upper semi-continuous border implicator, $\mathscr{N}$ an involutive negator and $\alpha \in [0,1]$. We will prove $\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}(\hat{\alpha}) = \hat{\alpha}$, as the statement for the upper approximation operator follows by duality. Since (INC) holds, it holds that $\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}(\hat{\alpha}) \subseteq \hat{\alpha}$. For the other inclusion, let $x \in U$ and $K^* \in \mathbb{C}$ such that $K^*(x) = 1$, then

$$\begin{aligned}
(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}(\hat{\alpha}))(x) &\geq \xi[\mathscr{I}](K^*(x), \inf_{y \in U} \mathscr{I}(K^*(y), \alpha)) \\
&= \xi[\mathscr{I}](1, \inf_{y \in U} \mathscr{I}(K^*(y), \alpha)) \\
&= \inf\{c \in [0,1] \mid \mathscr{I}(1, c) \geq \inf_{y \in U} \mathscr{I}(K^*(y), \alpha)\} \\
&= \inf\{c \in [0,1] \mid c \geq \inf_{y \in U} \mathscr{I}(K^*(y), \alpha)\} \\
&= \inf_{y \in U} \mathscr{I}(K^*(y), \alpha) \\
&\geq \inf_{y \in U} \mathscr{I}(1, \alpha) \\
&= \alpha.
\end{aligned}$$

We conclude that the pair $(\underline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I}}, \overline{\text{apr}}'_{\mathbb{C},\text{In},\mathscr{I},\mathscr{N}})$ satisfies (CS). □

8.1.3 Model of Wu et al.

The following model we discuss was introduced by Wu et al. [168]. It is inspired by the use of weak α-level sets for $K \in \mathbb{C}$.

Definition 8.1.6. [168] Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, then the pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ of fuzzy covering-based approximation operators is defined as follows: let $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x) &= \sup_{K\in\mathbb{C}} \inf\{X(y) \mid K(y) \geq K(x), y \in U\},\\ (\overline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x) &= \inf_{K\in\mathbb{C}} \sup\{X(y) \mid K(y) \geq K(x), y \in U\}.\end{aligned}$$

Note that this model does not use fuzzy logical connectives. We discuss its properties in the following proposition.

Proposition 8.1.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space.

- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ satisfies (D) with respect to an involutive negator $\mathscr{N}$.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ satisfies (INC), (SM), (ID), (CS) and (UE).

Proof. In [168], the properties (INC), (SM), (ID) and (UE) are proven. Moreover, it was proven that (D) is satisfied with respect to the standard negator. However, as every involutive negator $\mathscr{N}$ is continuous [4], it holds for every $X \in \mathscr{F}(U)$ and $x \in U$ that

$$\begin{aligned}(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X^{\mathscr{N}})^{\mathscr{N}})(x) &= \mathscr{N}\left(\sup_{K\in\mathbb{C}} \inf\{\mathscr{N}(X(y)) \mid K(y) \geq K(x), y \in U\}\right)\\ &= \inf_{K\in\mathbb{C}} \mathscr{N}(\inf\{\mathscr{N}(X(y)) \mid K(y) \geq K(x), y \in U\})\\ &= \inf_{K\in\mathbb{C}} \sup\{X(y) \mid K(y) \geq K(x), y \in U\}.\end{aligned}$$

To prove that the pair $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ satisfies (CS), let $\alpha \in [0,1]$ and $x \in U$, then

$$\begin{aligned}(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(\hat{\alpha}))(x) &= \sup_{K\in\mathbb{C}} \inf\{\alpha \mid K(y) \geq K(x), y \in U\}\\ &= \alpha.\end{aligned}$$

The equality $\overline{\text{apr}}'_{\mathbb{C},\text{Wu}}(\hat{\alpha}) = \hat{\alpha}$ can be proven similarly. □

Note that in [168], it is stated that $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}})$ also satisfies (IU). However, this is not correct, as illustrated in the next example.

Example 8.1.8. Let $U = \{x, y, z\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.7/y + 1/z$ and $K_2 = 0.8/x + 1/y + 1/z$. Consider the fuzzy sets $X = 0.6/x + 0/y + 0.3/z$ and $Y = 0.2/x + 0.8/y + 0.4/z$, then $X \cap Y = 0.2/x + 0/y + 0.3/z$ when the minimum operator is considered. We obtain that

$$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}(X) \cap \underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}(Y))(z) = \min(0.3, 0.4) = 0.3,$$

but $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}(X \cap Y))(z) = 0.2$. Hence, we conclude that the (IU) property does not hold.

8.1.4 Model induced by the theory of representation by levels

A possible way to construct a fuzzy extension of the crisp operator $\underline{\mathrm{apr}}'_{\mathbb{C}}$ is to apply the technique of representation by levels stated in Section 5.4. Note that we assume U and $\mathbb{C}$ to be finite, in order to induce a finite set of levels.

Definition 8.1.9. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $X \in \mathscr{F}(U)$. The fuzzy set $\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)$ is represented by the RL $(\Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}, \rho_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)})$, with

$$\begin{aligned} \Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)} &= \Lambda_X \cup \Lambda_{\mathbb{C}} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, m \in \mathbb{N} \setminus \{0\} \\ \rho_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}(\alpha) &= \bigcup \{K_\alpha \mid K \in \mathbb{C}, K_\alpha \subseteq X_\alpha\}, \end{aligned}$$

for all $\alpha \in \Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}$. To obtain the membership degree of x in $\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)$, we compute the fuzzy summary:

$$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X))(x) = \sum_{\{\alpha_i \in \Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)} \mid x \in \rho_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}(\alpha_i)\}} (\alpha_i - \alpha_{i+1}),$$

where we have ranked the elements of $\Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}$ as follows:

$$1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0.$$

The upper approximation operator $\overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}$ is obtained in a similar way, by taking

$$\rho_{\overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}(\alpha) = \mathrm{co}(\rho_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(\mathrm{co}(X))})$$

for each $\alpha \in \Lambda_{\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}(X)}$.

It is clear that for a crisp set X and a crisp covering $\mathbb{C}$, the pair of tight granule-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$ is obtained. Due to Proposition 5.4.1, this model satisfies all properties.

Proposition 8.1.10. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite.

- The pair $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}})$ satisfies (D) with respect to an involutive negator $\mathscr{N}$.
- The pair $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}})$ satisfies (INC), (SM), (ID), (CS) and (UE).

Proof. This follows immediately from Proposition 5.4.1. □

8.1.5 Model of intuitive extension

The final model extending the tight covering-based approximation operators we discuss is an intuitive extension of the crisp case. The lower approximation operator is obtained by replacing the union by the supremum, and by taking the membership degrees of x into account.

Definition 8.1.11. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{N}$ an involutive negator, then the pair of fuzzy covering-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx},\mathscr{N}})$ is defined as follows: let $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}(X))(x) &= \sup_{K\in\mathbb{C}}\{K(x) \mid K \subseteq X\},\\ (\overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx},\mathscr{N}}(X))(x) &= (\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{In},\mathscr{I}}(X^{\mathscr{N}}))^{\mathscr{N}}(x).\end{aligned}$$

It is clear that for a crisp covering $\mathbb{C}$ and a crisp set X the tight granule-based approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$ are obtained. A drawback of this model is that it is quite extreme: if for each $K \in \mathbb{C}$ it holds that $K \nsubseteq X$ for a given $X \in \mathscr{F}(U)$, then $\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}(X) = \emptyset$. In the following proposition, we discuss the properties of this model.

Proposition 8.1.12. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space and $\mathcal{N}$ an involutive negator.

- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}, \overline{\text{apr}}'_{\mathbb{C},\text{InEx},\mathcal{N}})$ satisfies (D) with respect to $\mathcal{N}$.
- The pair $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}, \overline{\text{apr}}'_{\mathbb{C},\text{InEx},\mathcal{N}})$ satisfies (INC), (SM), (ID) and (UE).

Proof. The duality property follows immediately by definition. Hence, we only discuss the other properties for the lower fuzzy approximation operator $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}$. It is clear that $\forall X \in \mathcal{F}(U)$ it holds that $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X) \subseteq X$, thus, (INC) holds. Moreover, for $X, Y \in \mathcal{F}(U)$ with $X \subseteq Y$, it holds that $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X) \subseteq \underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(Y)$. Hence, (SM) is satisfied. In addition, (ID) is satisfied since for all $K \in \mathbb{C}$ and $X \in \mathcal{F}(U)$ it holds that $K \subseteq \underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X)$ if and only if $K \subseteq X$. To end, for $x \in U$ it holds that

- $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(\emptyset))(x) = \sup\limits_{K\in\mathbb{C}} \emptyset = 0$,
- $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(U))(x) = \sup\limits_{K\in\mathbb{C}} K(x) = 1$.

We conclude that the pair $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}, \overline{\text{apr}}'_{\mathbb{C},\text{InEx},\mathcal{N}})$ satisfies (UE). □

Note that this model does not satisfy (CS) as illustrated in the next example.

Example 8.1.13. Let $U = \{x, y\}$ and $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.3/y$ and $K_2 = 0.3/x + 1/y$, and let $\alpha = 0.4$. The inclusions $K_1 \subseteq \hat{\alpha}$ and $K_2 \subseteq \hat{\alpha}$ do not hold. Hence, $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(\hat{\alpha}))(x) = 0$. We conclude that the property (CS) is not satisfied.

8.2 Fuzzy extensions of loose granule-based approximation operators

Before discussing different models extending the loose covering-based approximation operators defined in Eqs. (2.21) and (2.22), we recall Proposition 4.2.3: for a crisp covering $\mathbb{C}$ it holds that the pair of approximation operators $(\underline{\text{apr}}''_{\mathbb{C}}, \overline{\text{apr}}''_{\mathbb{C}})$ equals the pair of operators $(\underline{\text{apr}}_{N_4^{\mathbb{C}}}, \overline{\text{apr}}_{N_4^{\mathbb{C}}})$. This follows from a result concerning crisp approximation operators which has been obtained by Yao in [179]. A similar result for fuzzy approximation operators is given by Wu et al. [169].

Proposition 8.2.1. [169] Let $H\colon \mathscr{F}(U) \to \mathscr{F}(U)$ be a mapping and $\mathscr{T}$ a left-continuous t-norm. The operator H satisfies the following axioms:

$$(U1) \quad \forall X \in \mathscr{F}(U), \forall \alpha \in [0,1]\colon H(\hat{\alpha} \cap_{\mathscr{T}} X) = \hat{\alpha} \cap_{\mathscr{T}} H(X),$$
$$(U2) \quad \forall X_j \in \mathscr{F}(U), j \in J\colon H\left(\bigcup_{j\in J} X_j\right) = \bigcup_{j\in J} H(X_j),$$

if and only if there exists a fuzzy relation R on U such that $H = \overline{\text{apr}}_{R,\mathscr{T}}$.

Hence, if an operator on $\mathscr{F}(U)$ satisfies the axioms $(U1)$ and $(U2)$, then it is equivalent to a fuzzy element-based upper approximation operator on U. Moreover, the fuzzy relation R mentioned in the above proposition is defined by

$$\forall x, y \in U\colon R(x,y) = (H(1_x))(y). \tag{8.1}$$

8.2.1 Model of Li et al.

Besides defining a fuzzy extension of the tight covering-based approximation operators, Li et al. defined a fuzzy extension of the loose covering-based approximation operators.

Definition 8.2.2. [91] Let $(U, \mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{T}$ a t-norm and $\mathscr{I}$ an implicator, then the pair of fuzzy covering-based approximation operators $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ is defined as follows: let $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}(X))(x) &= \inf_{K\in\mathbb{C}} \mathscr{I}(K(x), \inf_{y\in U} \mathscr{I}(K(y), X(y))),\\ (\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(X))(x) &= \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), \sup_{y\in U} \mathscr{T}(K(y), X(y))).\end{aligned}$$

We prove that the upper approximation operator of this model is equivalent to an element-based one, when a left-continuous t-norm is taken into consideration.

Proposition 8.2.3. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space and $\mathscr{T}$ a left-continuous t-norm, then

$$\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}} = \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathscr{T}},$$

where $N_4^{\mathbb{C}}$ is defined with respect to $\mathscr{T}$.

Proof. In [91] it is proven that $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}$ satisfies axiom (U2). Moreover, we have for $\alpha \in [0,1]$, $X \in \mathscr{F}(U)$ and $x \in U$ that

$$\begin{aligned}
(\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(\hat{\alpha} \cap_{\mathscr{T}} X))(x) &= \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), \sup_{y\in U} \mathscr{T}(K(y), \mathscr{T}(\alpha, X(y)))) \\
&= \sup_{K\in\mathbb{C}} \sup_{y\in U} \mathscr{T}(K(x), \mathscr{T}(K(y), \mathscr{T}(\alpha, X(y)))) \\
&= \sup_{K\in\mathbb{C}} \sup_{y\in U} \mathscr{T}(\alpha, \mathscr{T}(K(x), \mathscr{T}(K(y), X(y)))) \\
&= \mathscr{T}(\alpha, \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), \sup_{y\in U} \mathscr{T}(K(y), X(y)))) \\
&= \mathscr{T}(\alpha, (\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(X))(x)) \\
&= (\hat{\alpha} \cap_{\mathscr{T}} \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(X))(x)
\end{aligned}$$

hence, $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}$ satisfies axiom (U1). By Proposition 8.2.1, $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}$ is equivalent with a fuzzy relation-based upper approximation operator. This fuzzy relation R is defined by, for $x, y \in U$,

$$\begin{aligned}
R(x,y) &= (\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(1_x))(y) \\
&= \sup_{K\in\mathbb{C}} \mathscr{T}(K(y), \sup_{z\in U} \mathscr{T}(K(z), 1_x(z))) \\
&= \sup_{K\in\mathbb{C}} \mathscr{T}(K(y), K(x)) \\
&= \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), K(y)) \\
&= N_4^{\mathbb{C}}(x)(y)
\end{aligned}$$

thus, we conclude that $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}} = \overline{\text{apr}}_{R,\mathscr{T}} = \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathscr{T}}$. □

The properties of $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ are given in the proposition below.

Proposition 8.2.4. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space, $\mathscr{T}$ a t-norm and $\mathscr{I}$ an implicator.

- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (D) with respect to the involutive negator $\mathscr{N}$ if $\mathscr{T}$ is an IMTL-t-norm, $\mathscr{I}$ is its R-implicator and $\mathscr{N}$ equals the negator induced by $\mathscr{I}$.
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (D) with respect to the involutive negator $\mathscr{N}$ if $\mathscr{I}$ is the S-implicator with respect to the t-conorm $\mathscr{S}$ and the negator $\mathscr{N}$, where $\mathscr{S}$ is the $\mathscr{N}$-dual of $\mathscr{T}$.

- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (INC) if $\mathscr{I}$ is a border implicator.
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (SM) and (IU).
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (CS) if $\mathscr{I}$ is a border implicator.
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (UE).
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (A) if $\mathscr{T}$ is a left-continuous t-norm and $\mathscr{I}$ is its R-implicator.
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (CM).

Proof. To prove that $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}})$ satisfies (A), let X and Y be fuzzy sets, $\mathscr{T}$ a left-continuous t-norm and $\mathscr{I}$ its R-implicator. We have

$$\begin{aligned}
& \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(X) \subseteq Y \\
\Leftrightarrow\quad & \forall x \in U: \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), \sup_{y\in U} \mathscr{T}(K(y), X(y))) \leq Y(x) \\
\Leftrightarrow\quad & \forall x \in U, \forall K \in \mathbb{C}: \mathscr{T}(\sup_{y\in U} \mathscr{T}(K(y), X(y)), K(x)) \leq Y(x) \\
\Leftrightarrow\quad & \forall x \in U, \forall K \in \mathbb{C}: \sup_{y\in U} \mathscr{T}(K(y), X(y)) \leq \mathscr{I}(K(x), Y(x)) \\
\Leftrightarrow\quad & \forall x \in U, \forall K \in \mathbb{C}, \forall y \in U: \mathscr{T}(K(y), X(y)) \leq \mathscr{I}(K(x), Y(x)) \\
\Leftrightarrow\quad & \forall K \in \mathbb{C}, \forall y \in U: \mathscr{T}(X(y), K(y)) \leq \inf_{x\in U} \mathscr{I}(K(x), Y(x)) \\
\Leftrightarrow\quad & \forall K \in \mathbb{C}, \forall y \in U: X(y) \leq \mathscr{I}(K(y), \inf_{x\in U} \mathscr{I}(K(x), Y(x))) \\
\Leftrightarrow\quad & \forall y \in U: X(y) \leq \inf_{K\in\mathbb{C}} \mathscr{I}(K(y), \inf_{x\in U} \mathscr{I}(K(x), Y(x))) \\
\Leftrightarrow\quad & X \subseteq \underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{I}}(Y).
\end{aligned}$$

To prove (CM), let $\mathbb{C}, \mathbb{C}'$ be two fuzzy coverings of U such that $\mathbb{C} \sqsubseteq \mathbb{C}'$. For $K \in \mathbb{C}$, denote $L_K \in \mathbb{C}'$ such that $K \subseteq L_K$. We prove the monotonicity for the upper approximation operator, for the lower approximation operator the proof is similar. Let $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}
(\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathscr{T}}(X))(x) &= \sup_{K\in\mathbb{C}} \mathscr{T}(K(x), \sup_{y\in U} \mathscr{T}(K(y), X(y))) \\
&\leq \sup_{K\in\mathbb{C}} \mathscr{T}(L_K(x), \sup_{y\in U} \mathscr{T}(L_K(y), X(y)))
\end{aligned}$$

$$\begin{aligned} &\leq \sup_{L\in\mathbb{C}'}\mathcal{T}(L(x),\sup_{y\in U}\mathcal{T}(L(y),X(y))) \\ &= (\overline{\text{apr}}''_{\mathbb{C}',\text{Li},\mathcal{T}}(X))(x) \end{aligned}$$

The other properties are proven in [91]. □

From the above proposition, we obtain that the equality

$$\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{I}} = \underline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{I}}$$

holds by duality if $\mathcal{T}$ is an IMTL-t-norm with $\mathcal{I}$ its R-implicator, or $\mathcal{I}$ is defined using the $\mathcal{N}$-dual t-conorm $\mathcal{S}$ of the left-continuous t-norm $\mathcal{T}$. Moreover, note the resemblance between Proposition 8.2.4 and the results obtained in Section 7.3, as $N_4^{\mathbb{C}}$ is a reflexive and symmetric fuzzy neighborhood operator.

8.2.2 Model induced by the theory of representation by levels

Another possible fuzzy extension of the loose covering-based approximation operators is constructed using representation by levels.

Definition 8.2.5. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $X \in \mathcal{F}(U)$. The fuzzy set $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)$ is represented by the RL $(\Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}, \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)})$, with

$$\begin{aligned} \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)} &= \Lambda_X \cup \Lambda_{\mathbb{C}} = \{\alpha_1,\alpha_2,\ldots,\alpha_m\}, m \in \mathbb{N}\setminus\{0\}, \\ \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\alpha) &= \bigcup\{K_\alpha \mid K \in \mathbb{C}, K_\alpha \cap X_\alpha \neq \emptyset\}, \end{aligned}$$

for all $\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}$. To obtain the membership degree of x in $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)$, we compute the fuzzy summary:

$$(\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(x) = \sum_{\{\alpha_i\in\Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)} \mid x\in\rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\alpha_i)\}} (\alpha_i - \alpha_{i+1}),$$

where we have ranked the elements of $\Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}$ as follows:

$$1 = \alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha_{m+1} = 0.$$

The lower approximation operator $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}$ is obtained in a similar way, by taking

$$\rho_{\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\alpha) = \text{co}(\rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(\text{co}(X))}(\alpha))$$

for each $\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}$.

Since the crisp representatives of the upper approximation operator are nested, we have the following characterization for the fuzzy upper approximation operator $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}$:

Proposition 8.2.6. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $X \in \mathscr{F}(U)$ and $x \in U$. Let $1 \leq k \leq m$ such that

$$\alpha_k \quad = \quad \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)} \mid x \in \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\alpha)\},$$

then $(\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(x) = \alpha_k$.

Proof. We first prove that $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)$ is represented by nested levels. Let β and γ be levels in $\Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}$ with $\beta \geq \gamma$ and assume $y \in \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\beta)$, then

$$y \in \bigcup\{K_\beta \mid K \in \mathbb{C}, K_\beta \cap X_\beta \neq \emptyset\}.$$

Let $K \in \mathbb{C}$ be such that $y \in K_\beta$ and $K_\beta \cap X_\beta \neq \emptyset$. Since $K(y) \geq \beta \geq \gamma$, $y \in K_\gamma$. Furthermore, $K_\gamma \cap X_\gamma \supseteq K_\beta \cap X_\beta \neq \emptyset$. Hence, $y \in \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\gamma)$ and thus, the crisp representation of $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)$ are nested. Therefore, we obtain for $x \in U$ that

$$\begin{aligned} & (\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(x) \\ = \; & (\alpha_k - \alpha_{k+1}) + (\alpha_{k+1} - \alpha_{k+2}) + \ldots + (\alpha_{m-1} - \alpha_m) + (\alpha_m - 0) \\ = \; & \alpha_k. \end{aligned}$$

□

Similarly as with the fuzzy loose upper approximation operator of Li et al., the upper approximation operator of this model is equivalent with an element-based one, when the minimum t-norm is considered.

Proposition 8.2.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite and $\mathcal{T}_M$ the minimum t-norm, then

$$\overline{\text{apr}}''_{\mathbb{C},\text{RBL}} = \overline{\text{apr}}_{N^{\mathbb{C}}_{4,\min},\mathcal{T}_M},$$

where $N^{\mathbb{C}}_{4,\min}$ denotes the fuzzy neighborhood operator $N^{\mathbb{C}}_4$ defined using the minimum t-norm.

Proof. By Proposition 5.4.1, $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}$ satisfies axiom (U1) with respect to $\mathcal{T}_M$ and axiom (U2), since for a crisp covering $\mathbb{C}$, $\overline{\text{apr}}''_{\mathbb{C}}$ satisfies the crisp equivalents of axioms (U1) and (U2). Furthermore, by Proposition 8.2.6 it holds for $x,y \in U$ that

$$\begin{aligned} R(x,y) &= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(1_x)} \mid y \in \rho_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(1_x)}(\alpha)\} \\ &= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(1_x)} \mid \exists K \in \mathbb{C}\colon K(y) \geq \alpha \wedge K(x) \geq \alpha\} \\ &= \max\{\alpha \in \Lambda_{\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(1_x)} \mid \sup_{K\in\mathbb{C}} \min(K(y),K(x)) \geq \alpha\} \\ &= \sup_{K\in\mathbb{C}} \min(K(y),K(x)) \\ &= \sup_{K\in\mathbb{C}} \min(K(x),K(y)) \\ &= N^{\mathbb{C}}_{4,\min}(x)(y) \end{aligned}$$

Thus, we conclude that $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}} = \overline{\text{apr}}_{R,\mathcal{T}_M} = \overline{\text{apr}}_{N^{\mathbb{C}}_{4,\min},\mathcal{T}_M}$. □

Corollary 8.2.8. When the minimum operator is used to define $N^{\mathbb{C}}_4$ and $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{T}}$, it holds that $\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{T}} = \overline{\text{apr}}''_{\mathbb{C},\text{RBL}}$. If another left-continuous t-norm $\mathcal{T}$ is used, we obtain that $(\overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{T}}(X))(x) \leq (\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(x)$ for each $X \in \mathcal{F}(U)$ and $x \in U$, as the minimum operator is the largest t-norm.

Note that $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}$ does not satisfy axiom (U1) with respect to every left-continuous t-norm, as illustrated in the next example.

Example 8.2.9. Let $U = \{x,y\}$ and $\mathbb{C} = \{K_1,K_2\}$ with $K_1 = 1/x + 0.3/y$ and $K_2 = 0.8/x + 1/y$. Let $X = 0.7/x + 0.8/y$ and $Y = 0.8/x + 0.4/y$ be fuzzy sets in U and let $\mathcal{T}_p$ be the product t-norm. Then

$$(\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X) \cap_{\mathcal{T}_p} \overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(Y))(x) = 0.8 \cdot 0.8 = 0.64$$

and $\overline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X \cap_{\mathcal{T}_p} Y)(x) = 0.56$.

Let $\mathscr{N}$ be an involutive negator, from the above proposition and Proposition 7.3.1, we derive that $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}} = \underline{\text{apr}}_{\text{N},\mathscr{I}}$, with $\text{N} = N^{\mathbb{C}}_{4,\min}$ and

$$\forall a, b \in [0,1]\colon \mathscr{I}(a,b) = \mathscr{N}(\min(a, \mathscr{N}(b))).$$

To end this section, we discuss the properties of this model.

Proposition 8.2.10. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite.

- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}, \overline{\text{apr}}''_{\mathbb{C},\text{RBL}})$ satisfies (D) with respect to an involutive negator $\mathscr{N}$.
- The pair $(\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}, \overline{\text{apr}}''_{\mathbb{C},\text{RBL}})$ satisfies (INC), (SM), (IU), (CS), (UE), (A) and (CM).

Proof. Follows immediately from Proposition 5.4.1. □

In the next section, we study equalities and partial order relations with respect to $\leq$ between the different fuzzy covering-based rough set models presented in Sections 8.1 and 8.2 and the IC model defined with different fuzzy neighborhood operators presented in Chapter 6.

8.3 Hasse diagram of fuzzy covering-based approximation operators

Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm, $\mathscr{I}$ its R-implicator and $\mathscr{N}$ the induced negator of $\mathscr{I}$. These parameters will be used for all the fuzzy neighborhood operators and fuzzy covering-based approximation operators we consider in this section. We will use an IMTL-t-norm to guarantee the duality property such that we only need to discuss partial order relations between the lower approximation operators. Moreover, since the fuzzy covering $\mathbb{C}$ is finite, $\mathscr{T}$ is left-continuous and $\mathscr{I}$ is its R-implicator, the results obtained in Figure 6.2 are valid. Furthermore, some results presented in this section make use of the properties of an IMTL-t-norm, its R-implicator and the induced negator which is involutive.

First, we discuss possible equalities between different models. In Chapter 6, 17 groups of fuzzy neighborhood operators are discussed. It is trivial to see that two equal fuzzy neighborhood operators yield the same pair of fuzzy neighborhoood-based approximation operators, i.e.,

$$\mathrm{N} = \mathrm{N}' \Rightarrow (\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}) = (\underline{\mathrm{apr}}_{\mathrm{N'},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N'},\mathscr{T}}).$$

Moreover, the inverse implication holds:

Proposition 8.3.1. Let N and N′ be two fuzzy neighborhood operators on U, then $\mathrm{N} = \mathrm{N}'$ if and only if $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}) = (\underline{\mathrm{apr}}_{\mathrm{N'},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N'},\mathscr{T}})$.

Proof. If $\mathrm{N} = \mathrm{N}'$, then $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}) = (\underline{\mathrm{apr}}_{\mathrm{N'},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N'},\mathscr{T}})$ holds trivially. On the other hand, let $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}) = (\underline{\mathrm{apr}}_{\mathrm{N'},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N'},\mathscr{T}})$ and let $x, y \in U$, then

$$\begin{aligned} & (\overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}}(1_y))(x) = (\overline{\mathrm{apr}}_{\mathrm{N'},\mathscr{T}}(1_y))(x) \\ \Rightarrow\quad & \mathscr{T}(\mathrm{N}(x)(y), 1) = \mathscr{T}(\mathrm{N}'(x)(y), 1) \\ \Rightarrow\quad & \mathrm{N}(x)(y) = \mathrm{N}'(x)(y). \end{aligned}$$

We conclude that $\mathrm{N} = \mathrm{N}'$. □

Hence, the 17 groups of fuzzy neighborhood operators discussed in Chapter 6 induce 17 different pairs of fuzzy neighborhood-based approximation operators $(\underline{\mathrm{apr}}_{\mathrm{N},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{N},\mathscr{T}})$.

Next, given the IMTL-t-norm $\mathscr{T}$ and its R-implicator $\mathscr{I}$, it holds that the tight granule-based model of Inuiguichi et al. defined in Definition 8.1.4 coincides with the tight granule-based model of Li et al. defined in Definition 8.1.1.

Proposition 8.3.2. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm and $\mathscr{I}$ its R-implicator, then

$$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Li},\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Li},\mathscr{T},\mathscr{I}}) = (\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{In},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{In},\mathscr{I},\mathscr{N}}).$$

Proof. For every left-continuous t-norm $\mathcal{T}$ it holds that $\xi[\mathcal{I}] = \mathcal{T}$ when we consider its R-implicator: let $a, b \in [0,1]$, then

$$\begin{aligned} \xi[\mathcal{I}](a,b) &= \inf\{c \in [0,1] \mid \mathcal{I}(a,c) \geq b\} \\ &= \inf\{c \in [0,1] \mid \mathcal{T}(a,b) \leq c\} \\ &= \mathcal{T}(a,b), \end{aligned}$$

by the residuation principle. Hence, $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}} = \underline{\text{apr}}'_{\mathbb{C},\text{In},\mathcal{I}}$. The other equality follows by duality with respect to the negator $\mathcal{N}$ induced by $\mathcal{I}$. □

Finally, recall that for an IMTL-t-norm $\mathcal{T}$ and its R-implicator $\mathcal{I}$ it holds that $(\underline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{I}}, \overline{\text{apr}}''_{\mathbb{C},\text{Li},\mathcal{T}}) = (\underline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{I}}, \overline{\text{apr}}_{N_4^{\mathbb{C}},\mathcal{T}})$ (see Proposition 8.2.3). Proposition 8.2.10 for the pair $(\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}, \overline{\text{apr}}''_{\mathbb{C},\text{RBL}})$ is not applicable, as the minimum operator is not an IMTL-t-norm. Hence, we will consider 22 different pairs of fuzzy covering-based approximation operators, presented in Table 8.1.

Next, we want to establish the Hasse diagram of the 22 lower fuzzy covering-based approximation operators with respect to the partial order $\leq$ on approximation operators. Since all pairs in Table 8.1 are dual, the Hasse diagram for the upper fuzzy covering-based approximation operators with respect to $\leq$ is immediately obtained from the Hasse diagram of the lower approximation operators.

The partial order relations of the first 17 lower approximation operators follows immediately from the results obtained in Figure 6.2, since the following proposition holds:

Proposition 8.3.3. Let N and N′ be two fuzzy neighborhood operators on U, then $\text{N} \preceq \text{N}'$ if and only if $(\underline{\text{apr}}_{\text{N}',\mathcal{I}}, \overline{\text{apr}}_{\text{N}',\mathcal{T}}) \leq (\underline{\text{apr}}_{\text{N},\mathcal{I}}, \overline{\text{apr}}_{\text{N},\mathcal{T}})$.

Proof. Let $\text{N} \preceq \text{N}'$, then $(\underline{\text{apr}}_{\text{N}',\mathcal{I}}, \overline{\text{apr}}_{\text{N}',\mathcal{T}}) \leq (\underline{\text{apr}}_{\text{N},\mathcal{I}}, \overline{\text{apr}}_{\text{N},\mathcal{T}})$ since the neighborhood monotonicity property is satisfied. On the other hand, if

$$(\underline{\text{apr}}_{\text{N}',\mathcal{I}}, \overline{\text{apr}}_{\text{N}',\mathcal{T}}) \leq (\underline{\text{apr}}_{\text{N},\mathcal{I}}, \overline{\text{apr}}_{\text{N},\mathcal{T}}),$$

then for $x, y \in U$ it holds that:

$$\text{N}(x)(y) = (\overline{\text{apr}}_{\text{N},\mathcal{T}}(1_y))(x) \leq (\overline{\text{apr}}_{\text{N}',\mathcal{T}}(1_y))(x) = \text{N}'(x)(y).$$

Table 8.1: Fuzzy covering-based rough set models in the fuzzy covering approximation space $(U, \mathbb{C})$ with U and $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm, $\mathscr{I}$ its R-implicator and $\mathscr{N}$ the induced negator

No.	N	Pairs	No.	N	Pairs
1	$a1$	$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}},\mathscr{T}})$,	11	i	$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}_4},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}_4},\mathscr{T}})$
		$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}_1},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}_1},\mathscr{T}})$,	12	j_1	$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}},\mathscr{T}})$,
		$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}_3},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}_3},\mathscr{T}})$,			$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}_2},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}_2},\mathscr{T}})$,
		$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}_\cap},\mathscr{T}})$			$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}_\cap},\mathscr{T}})$,
2	$a2$	$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}_3},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}_3},\mathscr{T}})$			$(\underline{\mathrm{apr}}''_{\mathbb{C},\mathrm{Li},\mathscr{I}}, \overline{\mathrm{apr}}''_{\mathbb{C},\mathrm{Li},\mathscr{T}})$
3	b	$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}_3},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}_3},\mathscr{T}})$	13	j_2	$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}_2},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}_2},\mathscr{T}})$
4	c	$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}},\mathscr{T}})$,	14	k	$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}_4},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}_4},\mathscr{T}})$
		$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}_1},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}_1},\mathscr{T}})$	15	l	$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}_4},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}_4},\mathscr{T}})$
5	d	$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}_1},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}_1},\mathscr{T}})$	16	m	$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}_4},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}_4},\mathscr{T}})$
6	e	$(\underline{\mathrm{apr}}_{N_2^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\mathrm{apr}}_{N_2^{\mathbb{C}_\cap},\mathscr{T}})$	17	$N_{1,\mathrm{Ma}}^{\mathbb{C}}$	$(\underline{\mathrm{apr}}_{N_{1,\mathrm{Ma}}^{\mathbb{C}},\mathscr{I}}, \overline{\mathrm{apr}}_{N_{1,\mathrm{Ma}}^{\mathbb{C}},\mathscr{T}})$
7	f_1	$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}},\mathscr{T}})$,	18		$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Li},\mathscr{T},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Li},\mathscr{T},\mathscr{I}})$,
		$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}_2},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}_2},\mathscr{T}})$,			$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{In},\mathscr{I}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{In},\mathscr{I},\mathscr{N}})$
		$(\underline{\mathrm{apr}}_{N_3^{\mathbb{C}_\cap},\mathscr{I}}, \overline{\mathrm{apr}}_{N_3^{\mathbb{C}_\cap},\mathscr{T}})$	19		$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}})$
8	f_2	$(\underline{\mathrm{apr}}_{N_1^{\mathbb{C}_2},\mathscr{I}}, \overline{\mathrm{apr}}_{N_1^{\mathbb{C}_2},\mathscr{T}})$	20		$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{RBL}})$
9	g	$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}_3},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}_3},\mathscr{T}})$	21		$(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}, \overline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx},\mathscr{N}})$
10	h	$(\underline{\mathrm{apr}}_{N_4^{\mathbb{C}_1},\mathscr{I}}, \overline{\mathrm{apr}}_{N_4^{\mathbb{C}_1},\mathscr{T}})$	22		$(\underline{\mathrm{apr}}''_{\mathbb{C},\mathrm{RBL}}, \overline{\mathrm{apr}}''_{\mathbb{C},\mathrm{RBL}})$

We conclude that $\mathrm{N} \preceq \mathrm{N}'$. □

From the previous proposition we obtain that smaller fuzzy neighborhood operators yield more accurate fuzzy neighborhood-based approximation operators, i.e., they yield larger lower approximations and smaller upper approximations. Hence, the Hasse diagram for the lower approximation operators of pairs 1 – 17 in Table 8.1 can be found in Figure 8.1.

We now want to add the four fuzzy-covering based tight lower approximation operators to Figure 8.1. First, note that the following partial order relations between the four tight lower approximation operators hold:

Proposition 8.3.4. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, then

(a) $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{RBL}}$,

(b) $\underline{\text{apr}}'_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{Wu}}$,

(c) $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}$ for a left-continuous t-norm and its R-implicator,

(d) $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{Wu}}$ for a left-continuous t-norm and its R-implicator.

Proof. Let $X \in \mathscr{F}(U)$ and $x \in U$.

(a) If there is no $K \in \mathbb{C}$ with $K \subseteq X$, then $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X) = \emptyset$, thus the inclusion holds.

On the other hand, take $K \in \mathbb{C}$ with $K \subseteq X$, then for all $\alpha \in [0,1]$, $K_\alpha \subseteq X_\alpha$. We need to prove that $K(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x)$.

If $K(x) = 0$, then the inclusion holds, so assume $K(x) = \gamma$ with $\gamma \neq 0$. For all $\alpha \leq \gamma$ it holds that $K(x) \geq \alpha$, hence $x \in K_\alpha$. Since $K_\alpha \subseteq X_\alpha$, we obtain that $x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha)$.

Let $\Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $\alpha_i > \alpha_{i+1}$ for all $1 \leq i \leq m$ and $\alpha_{m+1} = 0$. Since $K(x) = \gamma$, $\gamma \in \Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}$. Therefore, there exists a $1 \leq i \leq m$

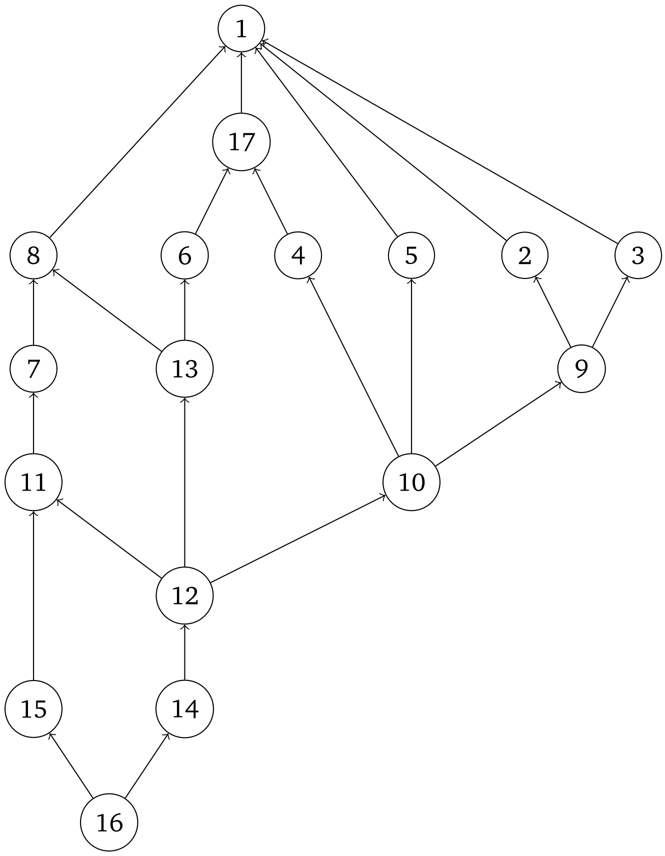


Figure 8.1: Hasse diagram of the first 17 fuzzy covering-based lower approximation operators presented in Table 8.1 for the fuzzy covering approximation space $(U, \mathbb{C})$ with $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm and $\mathscr{I}$ its R-implicator

such that $\gamma = \alpha_i$. By the fact that $x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha)$ for all $\alpha \leq \gamma$ and the definition of fuzzy summary, we obtain that

$$\begin{array}{rl} & (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) \\ \geq & (\alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - \alpha_{i+2}) + \ldots + (\alpha_m - \alpha_{m+1}) \\ = & \alpha_i \\ = & \gamma. \end{array}$$

Therefore, $K(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x)$, and thus,

$$(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X))(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x).$$

(b) If there is no $\alpha \in \Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}$ such that $x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha)$, then

$$(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) = 0$$

and thus, $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x)$.

On the other hand, let $\alpha^* \in \Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}$ be the largest level such that x belongs to $\rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha^*)$. We will prove that $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) \leq \alpha^*$ and $\alpha^* \leq (\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x)$.

For the first inequality, note that $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x)$ reaches its maximum if $x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha)$ for all $\alpha \leq \alpha^*$, i.e., if x is in all crisp representatives for the levels $\alpha \leq \alpha^*$. Denote $\Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $\alpha_i > \alpha_{i+1}$ for all $1 \leq i \leq m$ and $\alpha_{m+1} = 0$. Then there exists a $1 \leq i \leq m$ such that $\alpha^* = \alpha_i$. Hence,

$$\begin{array}{rl} & (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) \\ \leq & (\alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - \alpha_{i+2}) + \ldots + (\alpha_m - \alpha_{m+1}) \\ = & \alpha_i \\ = & \alpha^*. \end{array}$$

For the second inequality, since $x \in \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha^*)$, there exists a $K^* \in \mathbb{C}$ with $K^*_{\alpha^*} \subseteq X_{\alpha^*}$ and $x \in K^*_{\alpha^*}$. Moreover, for every $y \in U$ with $K^*(y) \geq K^*(x)$, it holds that $y \in K^*_{\alpha^*}$, and thus $y \in X_{\alpha^*}$. Therefore,

$$\begin{array}{rcl}(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x) & \geq & \inf\limits_{y\in U}\{X(y) \mid K^*(y) \geq K^*(x)\} \\ & \geq & \inf\limits_{y\in U}\{\alpha^* \mid K^*(y) \geq K^*(x)\} \\ & = & \alpha^*.\end{array}$$

We conclude that $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x)$.

(c) Let $\mathscr{T}$ be left-continuous and $\mathscr{I}$ its R-implicator. Define the set

$$\mathbb{C}' = \{K \in \mathbb{C} \mid K \subseteq X\} \subseteq \mathbb{C},$$

then for all $K \in \mathbb{C}'$ it holds that $\inf\limits_{y\in U}\mathscr{I}(K(y), X(y)) = 1$ (see [158]). Therefore, we obtain for $x \in U$ that

$$\begin{array}{rcl}(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))(x) & \geq & \sup\limits_{K\in\mathbb{C}'}\mathscr{T}(K(x), \inf\limits_{y\in U}\mathscr{I}(K(y), X(y))) \\ & = & \sup\limits_{K\in\mathbb{C}'}\mathscr{T}(K(x), 1) \\ & = & \sup\{K(x) \mid K \in \mathbb{C}'\} \\ & = & \sup\{K(x) \mid K \in \mathbb{C}, K \subseteq X\} \\ & = & (\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X))(x).\end{array}$$

We conclude that $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X))(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))(x)$.

(d) Let $\mathscr{T}$ be left-continuous and $\mathscr{I}$ its R-implicator. Let $K \in \mathbb{C}$ and denote $K_{K(x)} = \{y \in U \mid K(y) \leq K(x)\}$. We have that

$$\begin{array}{rcl}\mathscr{T}(K(x), \inf\limits_{y\in U}\mathscr{I}(K(y), X(y))) & \leq & \mathscr{T}(K(x), \inf\limits_{y\in K_{K(x)}}\mathscr{I}(K(y), X(y))) \\ & \leq & \mathscr{T}(K(x), \inf\limits_{y\in K_{K(x)}}\mathscr{I}(K(x), X(y))) \\ & = & \mathscr{T}(K(x), \mathscr{I}(K(x), \inf\limits_{y\in K_{K(x)}} X(y))) \\ & \leq & \inf\limits_{y\in K_{K(x)}} X(y),\end{array}$$

where we have used various properties which hold for a left-continuous t-norm and its R-implicator. As this holds for every $K \in \mathbb{C}$, we obtain that $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))(x) \leq (\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X))(x)$. □

As we assumed $\mathcal{T}$ to be an IMTL-t-norm, all partial order relations in Proposition 8.3.4 hold. Furthermore, by duality, we can obtain the partial order relations for the four fuzzy covering-based tight upper approximation operators if necessary.

Proposition 8.3.4 contains the only partial order relations which hold between these four fuzzy covering-based lower approximation operators as illustrated in the next example.

Example 8.3.5. Let $U = \{x, y, z\}$, let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ its R-implicator. Consider the fuzzy covering $\mathbb{C} = \{K_1, K_2, K_3\}$ with

$$K_1 = 0.6/x + 0.6/y + 0.6/z,$$
$$K_2 = 1/x + 1/y + 1/z,$$
$$K_3 = 0.9/x + 0.8/y + 0.6/z/$$

Let $X = 0.9/x + 0.7/y + 0.6/z$, then

- $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X) = 0.6/x + 0.6/y + 0.6/z$,
- $\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X) = \underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X) = 0.8/x + 0.7/y + 0.6/z$,
- $\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}(X) = 0.9/x + 0.7/y + 0.6/z$.

To illustrate that the model of Li et al. and the model induced by the theory of representation by levels are incomparable with each other, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.6/z$ and $K_2 = 0/x + 1/y + 1/z$, and $X = 1/x + 0/y + 0.4/z$, then $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) = 0.8$ and $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) = 0.4$, when the nilpotent minimum and its R-implicator is used. On the other hand, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0.7/y + 0.7/z$ and $K_2 = 0/x + 1/y + 1/z$, and $X = 0.5/x + 0.5/y + 0.4/z$, then $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) = 0.4$ and $(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) = 0.5$, when the Łukasiewicz t-norm and implicator are used.

In order to add the four fuzzy covering-based tight approximation operators to Figure 8.1, we discuss their comparability with respect to $\leq$ to the 17 fuzzy neighborhood-based lower approximation operators. However, note that for a crisp covering $\mathbb{C}$, the approximation operator $\underline{\text{apr}}'_{\mathbb{C}}$ is incomparable to the approximation operators $\underline{\text{apr}}_{\text{N}}$ with $\text{N} \in \{b, d, f_1, f_2, g, i, l\}$, hence, the four fuzzy covering-based tight approximation operators are incomparable with these seven fuzzy neighborhood-based approximation operators. Moreover, we only need to discuss the partial order relations which hold for a crisp covering. Thus, we need to study the following partial order relations for

$$\underline{\text{apr}} \in \{\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}, \underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \underline{\text{apr}}'_{\mathbb{C},\text{RBL}}, \underline{\text{apr}}'_{\mathbb{C},\text{InEx}}\}:$$

- $\underline{\text{apr}} \leq \underline{\text{apr}}_{\text{N},\mathcal{I}}$ for $\text{N} \in \{a1, a2, N^{\mathbb{C}}_{1,\text{Ma}}\}$,
- $\underline{\text{apr}}_{\text{N},\mathcal{I}} \leq \underline{\text{apr}}$ for $\text{N} \in \{c, e, h, j_1, j_2, k, m, N^{\mathbb{C}}_{1,\text{Ma}}\}$.

The following partial order relations hold:

Proposition 8.3.6. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathcal{T}$ an IMTL-t-norm and $\mathcal{I}$ its R-implicator, then

(a) $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}} \leq \underline{\text{apr}}_{N^{\mathbb{C}}_1,\mathcal{I}}$,

(b) $\underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathcal{I}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}$.

Proof. Let $X \in \mathcal{F}(U)$ and $x \in U$.

(a) First note that $\forall a, b, c \in [0, 1]$: $\mathcal{T}(a, \mathcal{I}(b, c)) \leq \mathcal{I}(\mathcal{I}(a, b), c)$. We have that

$$\begin{aligned}
(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) &= \sup_{K\in\mathbb{C}} \mathcal{T}(K(x), \inf_{y\in U} \mathcal{I}(K(y), X(y))) \\
&= \sup_{K\in\mathbb{C}} \inf_{y\in U} \mathcal{T}(K(x), \mathcal{I}(K(y), X(y))) \\
&\leq \sup_{K\in\mathbb{C}} \sup_{y\in U} \mathcal{I}(\mathcal{I}(K(x), K(y)), X(y)) \\
&= \sup_{K\in\mathbb{C}} \mathcal{I}\left(\inf_{y\in U} \mathcal{I}(K(x), K(y)), X(y)\right) \\
&= \sup_{K\in\mathbb{C}} \mathcal{I}\left(N^{\mathbb{C}}_1(x)(y), X(y)\right) \\
&= (\underline{\text{apr}}_{N^{\mathbb{C}}_1,\mathcal{I}}(X))(x)
\end{aligned}$$

(b) It holds that

$$(\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathcal{T},\mathcal{I}}(X))(x) \geq \sup_{K\in\mathbb{C},K(x)=1} \mathcal{T}(K(x), \inf_{y\in U} \mathcal{I}(K(y), X(y)))$$

$$
\begin{aligned}
&= \sup_{K\in\mathbb{C},K(x)=1} \inf_{y\in U} \mathcal{I}(K(y),X(y)) \\
&= \inf_{y\in U} \mathcal{I}\left(\inf_{K\in\mathbb{C},K(x)=1} K(y),X(y)\right) \\
&= \inf_{y\in U} \mathcal{I}(N^{\mathbb{C}}_{1,\mathrm{Ma}}(x)(y),X(y)) \\
&= (\underline{\mathrm{apr}}_{N^{\mathbb{C}}_{1,\mathrm{Ma}},\mathcal{I}}(X))(x)
\end{aligned}
$$

□

Note that Proposition 8.3.6 also holds for a left-continuous t-norm $\mathcal{T}$ and its R-implicator $\mathcal{I}$. By transitivity of the partial order relations $\leq$ we also have the following results:

Corollary 8.3.7. Let $(U,\mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathcal{T}$ an IMTL-t-norm and $\mathcal{I}$ its R-implicator, then

(a) $\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}} \leq \underline{\mathrm{apr}}_{N^{\mathbb{C}}_1,\mathcal{I}}$,

(b) $\underline{\mathrm{apr}}_{\mathrm{N},\mathcal{I}} \leq \underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Li},\mathcal{T},\mathcal{I}}$ for $\mathrm{N} \in \{c,e,h,j_1,j_2,k,m\}$,

(c) $\underline{\mathrm{apr}}_{\mathrm{N},\mathcal{I}} \leq \underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}$ for $\mathrm{N} \in \{c,e,h,j_1,j_2,k,m,N^{\mathbb{C}}_{1,\mathrm{Ma}}\}$.

In the following example we illustrate that no other partial order relations hold.

Example 8.3.8. Let $U = \{x,y,z\}$ and let $\mathbb{C} = \{K_1,K_2\}$ with $K_1 = 1/x+0/y+0.5/z$ and $K_2 = 0/x+1/y+1/z$. Let $\mathcal{T}$ be the Łukasiewicz t-norm and $\mathcal{I}$ the Łukasiewicz implicator.

(a) Let $X = 1/x+0.5/y+0/z$, then

- $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}(X))(x) = 0$,
- $(\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}}(X))(x) = 1$,
- $(\underline{\mathrm{apr}}_{\mathrm{N},\mathcal{I}}(X))(x) = 0.5$ for $\mathrm{N} \in \{a_1,a_2,c,e,h,j_1,j_2,k,m,N^{\mathbb{C}}_{1,\mathrm{Ma}}\}$.

We conclude that $\underline{\mathrm{apr}}_{\mathrm{N},\mathcal{I}} \leq \underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{InEx}}$ for $\mathrm{N} \in \{c,e,h,j_1,j_2,k,m,N^{\mathbb{C}}_{1,\mathrm{Ma}}\}$ and $\underline{\mathrm{apr}}'_{\mathbb{C},\mathrm{Wu}} \leq \underline{\mathrm{apr}}_{\mathrm{N},\mathcal{I}}$ for $\mathrm{N} \in \{a_1,a_2,N^{\mathbb{C}}_{1,\mathrm{Ma}}\}$ do not hold.

(b) Let $X = 1/x + 0/y + 0.7/z$, then it holds that

- $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X))(z) = (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(z) = (\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))(z) = 0.5$,
- $(\underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathscr{I}}(X))(z) = 0$.

We conclude that the following partial order relations do not hold:

(i) $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}} \leq \underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathscr{I}}$,

(ii) $\underline{\text{apr}}'_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathscr{I}}$,

(iii) $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}} \leq \underline{\text{apr}}_{N^{\mathbb{C}}_{1,\text{Ma}},\mathscr{I}}$

Next, consider $\mathbb{C} = \{K_1, K_2, K_3\}$ with

$$\begin{aligned} K_1 &= 1/x + 0.8/y + 0.6/z, \\ K_2 &= 0.2/x + 1/y + 0.6/z, \\ K_3 &= 0.2/x + 0.8/y + 1/z \end{aligned}$$

and let $\mathscr{T}$ be the Łukasiewicz t-norm and $\mathscr{I}$ the Łukasiewicz implicator.

(a) Let $X = 0.1/x + 0.5/y + 0.4/z$, then we have that

- $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(z) = 0.3$,
- $(\underline{\text{apr}}_{\text{N},\mathscr{I}}(X))(z) = 0.4$ for $\text{N} \in \{c, e, h, j_1, j_2, k, m, N^{\mathbb{C}}_{1,\text{Ma}}\}$.

We conclude that $\underline{\text{apr}}_{\text{N},\mathscr{I}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{RBL}}$ does not hold for the fuzzy neighborhood operators $\text{N} \in \{c, e, h, j_1, j_2, k, m, N^{\mathbb{C}}_{1,\text{Ma}}\}$.

(b) Let $X = 1/x + 1/y + 0.7/z$, then it holds that

- $(\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X))(y) = (\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(y) = (\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}}(X))(y) = 1$,
- $(\underline{\text{apr}}_{N^{\mathbb{C}_3}_2,\mathscr{I}}(X))(y) = 0.9$.

We conclude that the following partial order relations do not hold:

(i) $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}} \leq \underline{\text{apr}}_{N^{\mathbb{C}_3}_2,\mathscr{I}}$,

(ii) $\underline{\text{apr}}'_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}_{N^{\mathbb{C}_3}_2,\mathscr{I}}$,

(iii) $\underline{\text{apr}}'_{\mathbb{C},\text{Li},\mathscr{T},\mathscr{I}} \leq \underline{\text{apr}}_{N_2^{\mathbb{C}_3},\mathscr{I}}$.

Finally, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x + 0/y + 0.6/z$ and $K_2 = 0/x + 1/y + 1/z$. Let $\mathscr{T}$ be the nilpotent minimum and $\mathscr{I}$ its R-implicator. Let $X = 1/x + 0/y + 0.4/z$, then it holds that $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(x) = 0.8$ and $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}(X))(x) = 0.4$. Hence, $\underline{\text{apr}}'_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}$ does not hold.

To end, we only need to discuss the fuzzy covering-based loose lower approximation operator $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}$. Since for a crisp covering $\mathbb{C}$ it holds that $\underline{\text{apr}}''_{\mathbb{C}} \leq \underline{\text{apr}}'_{\mathbb{C}}$, we have the following property:

Proposition 8.3.9. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm and $\mathscr{I}$ its R-implicator, then $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{RBL}}$.

Proof. Let $X \in \mathscr{F}(U)$, then $\Lambda_{\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)} = \Lambda_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}$ and for every level $\alpha \in \Lambda_{\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}$ it holds that $\rho_{\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X)}(\alpha) \subseteq \rho_{\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X)}(\alpha)$. □

By the transitivity of $\leq$, we also have the following result:

Corollary 8.3.10. Let $(U, \mathbb{C})$ be a fuzzy covering approximation space with U and $\mathbb{C}$ finite, $\mathscr{T}$ an IMTL-t-norm and $\mathscr{I}$ its R-implicator, then $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}'_{\mathbb{C},\text{Wu}}$.

As illustrated in the next example, these are the only partial order relations which hold:

Example 8.3.11. Let $U = \{x, y, z\}$ and let $\mathbb{C} = \{K_1, K_2, K_3\}$ be a fuzzy covering on U with $K_1 = 1/x + 0.8/y + 0.6/z$, $K_2 = 0.2/x + 1/y + 0.6/z$ and $K_3 = 0.2/x + 0.8/y + 1/z$ and let $\mathscr{T}$ be the Łukasiewicz t-norm and $\mathscr{I}$ the Łukasiewicz implicator.

(a) Let $X = 0.1/x + 0.5/y + 0.4/z$, then we have that

- $(\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(y) = 0.1$,
- $(\underline{\text{apr}}'_{\mathbb{C},\text{RBL}}(X))(y) = (\underline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathscr{I}}(X))(y) = 0.3$.

We conclude that $\underline{\text{apr}} \leq \underline{\text{apr}}''_{\mathbb{C},\text{RBL}}$ does not hold for any of the first 20 lower approximation operators of Table 8.1.

(b) Let $X = 1/x + 1/y + 0.7/z$, then it holds that

- $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}(X) = 1/x + 1/y + 0.6/z$,
- $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}} = 1/x + 0.9/y + 0.7/z$.

We conclude that $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}$ and $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}$ are incomparable.

Futhermore, let $\mathbb{C} = \{K_1, K_2\}$ with $K_1 = 1/x+0/y+0.6/z$, $K_2 = 0/x+1/y+1/z$. Let $\mathcal{T}$ be the nilpotent minimum and $\mathcal{I}$ its R-implicator. Let $X = 1/x+0/y+0.4/z$, then it holds that $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{I}}(X))(x) = 0.4$ and $(\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}(X))(x) = 0.8$. Hence, $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}} \leq \underline{\text{apr}}$ does not hold for any of the first 18 lower approximation operators of Table 8.1.

Hence, the Hasse diagram with respect to $\leq$ representing all fuzzy covering-based lower approximation operators stated in Table 8.1 is given in Figure 8.2. Minimal elements of the Hasse diagram are given by the fuzzy covering-based lower approximation operators $\underline{\text{apr}}_{N_4^{\mathbb{C}_4},\mathcal{I}}$, $\underline{\text{apr}}'_{\mathbb{C},\text{InEx}}$ and $\underline{\text{apr}}''_{\mathbb{C},\text{RBL}}$. The fuzzy covering-based lower approximation operators $\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathcal{I}}$ and $\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}$ are maximal elements of the Hasse diagram. Therefore, these approximation operators provide the most accurate approximations.

8.4 Conclusions and future work

In this chapter, we have studied fuzzy covering-based rough set models which extend the tight and loose granule-based approximation operators. We have recalled three existing models and introduced two new ones which extend the tight approximation operators. Moreover, we recalled one model and introduced one model which extend the loose approximation operators. Both models are equivalent to fuzzy neighborhood-based models. For each of the seven models, we have discussed its properties. All models maintain the properties of the tight, respectively loose, approximation operators given some conditions on the used t-norm and implicator. Only the intuitive extension of the tight approximation operators does not satisfy the (CS) property.

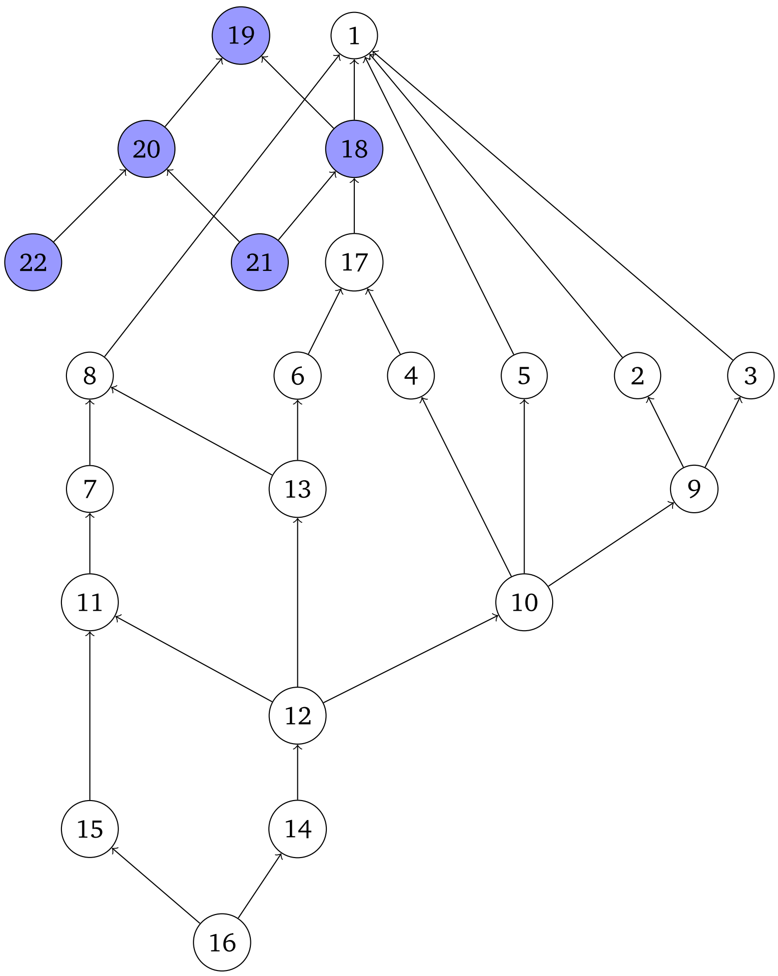


Figure 8.2: Hasse diagram of the fuzzy covering-based lower approximation operators presented in Table 8.1 for the fuzzy covering approximation space $(U, \mathbb{C})$ with $\mathbb{C}$ finite, $\mathcal{T}$ an IMTL-t-norm and $\mathcal{I}$ its R-implicator

Secondly, we have discussed partial order relations with respect to $\leq$ for a finite universe U, a finite fuzzy covering $\mathbb{C}$ on U, an IMTL-t-norm $\mathscr{T}$, its R-implicator $\mathscr{I}$ and its induced negator $\mathscr{N}$. We have studied 22 pairs of fuzzy covering-based rough set models, of which 17 pairs are based on a fuzzy neighborhood operator. The Hasse diagram of the lower approximation operators is presented in Figure 8.2. By duality, the Hasse diagram of the upper approximation operators is obtained by inverting Figure 8.2. Hence, we conclude that the pairs $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{T}})$ and $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ provide the most accurate approximations.

Future work includes the study of fuzzy extensions of other pairs of covering-based rough set approximation operators, e.g., the approximation operators of the framework of Yang and Li. In addition, we want to study the comparability between the approximation operators studied in this chapter and other fuzzy rough set approximation operators, such as the noise-tolerant approximation operators presented in Chapter 9. Finally, a future research direction is the application of fuzzy covering-based rough set models in feature and instance selection.

CHAPTER 9

Noise-tolerant fuzzy rough set models

The implicator-conjunctor-based fuzzy rough set model allows for a lot of flexibility in terms of the choice of the fuzzy logical connectives and the fuzzy relation or fuzzy neighborhood operator. However, the use of the infimum and supremum in Eqs. (7.1) and (7.2) limits their practical use, in a similar way as the $\forall$- and $\exists$-quantifiers restrict the application potential of Pawlak's original rough set model.

The core of the problem is that the result of the approximations is determined by a single best (sup) or worst (inf) element. This can be a disadvantage in a data analysis context, since data samples may be erroneous. Such noisy data can perturb the approximations and therefore weaken the machine learning algorithms which invoke them [72].

To address this problem, many authors have defined robust alternatives for the lower and upper approximation operators in fuzzy rough set theory, in a similar way as the VPRS model defined in Section 2.3 provides a noise-tolerant alternative for Pawlak's rough set definition.

In this chapter we discuss different robust fuzzy rough set models which have been proposed in literature. In Section 9.1, we study models which are frequency-based [16, 69, 70, 115, 116, 178], analogous to the VPRS model of Ziarko [203]. Another model adjusts the set which is approximated [196], which we discuss in Section 9.2. Moreover, in Section 9.3 we study models which use other aggregation operators than the infimum and supremum operators [19, 47]. For each considered model we discuss the specific criteria it uses for harnessing the approximations against noise. We generalize, correct or simplify its definition, without harming the original ideas it is based on. We study relationships which exist between the models, and also evaluate which properties of Pawlak's rough set model can be maintained. Similarly as for the VPRS model, making the models more flexible towards noise typically involves sacrificing some of the desirable properties they satisfy. Additionally, in Section 9.4 we will evaluate the claim whether the noise-tolerant models are more robust than the IC model from a practical perspective, by examining how stable their approximations are when the data are contaminated by noise. We conclude and state future work in Section 9.5.

Note that in this chapter we will consider the IC model based on a general fuzzy relation R, as the noise-tolerant models discussed here are defined with respect to a fuzzy relation. Moreover, we will assume the universe U to be finite, since the number of data samples, i.e., objects in U, in a real information system is always finite.

9.1 Noise-tolerant models based on frequency

We start with discussing frequency-based fuzzy rough set models. Models of this type are the variable precision fuzzy rough set model of Mieskowicz-Rolka and Rolka [115, 116], the vaguely quantified fuzzy rough set model of Cornelis et al. [16], the soft fuzzy rough set model of Hu et al. [69, 70] and the variable precision fuzzy rough set model based on fuzzy granules of Yao et al. [178]. The idea behind this type of robust models is that only a subset of the fuzzy set of predecessors $R^p(x)$ of an object $x \in U$ is taken into account when computing the lower and upper approximation in x, which is the same objective as the variable precision rough set model of Ziarko [203].

We start with the model of Mieszkowicz-Rolka and Rolka.

9.1.1 Variable precision fuzzy rough set model

Mieszkowicz-Rolka and Rolka [115] proposed the Variable Precision Fuzzy Rough Set (VPFRS) model in 2004, and later appeared in slightly revised form in [116]. They intended to design a robust fuzzy rough set model that covers both the seminal fuzzy rough set approach of Dubois and Prade [41, 42] and the VPRS model of Ziarko [84, 203].

We start by recalling some preliminary notions introduced by the authors, in order to define the VPFRS model.

Definition 9.1.1. [116] Given a WCP-implicator $\mathscr{I}$, a t-norm $\mathscr{T}$ and fuzzy sets X, Y in U, the *implication-based inclusion set* $\text{Incl}_{\mathscr{I}}(X,Y)$ *of* X *in* Y is defined by

$$\forall x \in U\colon\ \text{Incl}_{\mathscr{I}}(X,Y)(x) = \mathscr{I}(X(x), Y(x)) \tag{9.1}$$

and the *t-norm-based inclusion set* $\text{Incl}_{\mathscr{T}}(X,Y)$ *of* X *in* Y is defined by

$$\forall x \in U\colon\ \text{Incl}_{\mathscr{T}}(X,Y)(x) = \mathscr{T}(X(x), Y(x)). \tag{9.2}$$

Note that the name 't-norm-based inclusion set' is slightly misleading because this definition is not actually related to fuzzy inclusion. However, we adopt the terminology of [116]. Moreover, in [116], the authors defined $\text{Incl}_{\mathscr{I}}$ as follows: let $X, Y \in \mathscr{F}(U)$ and $x \in U$, then

$$\text{Incl}_{\mathscr{I}}(X,Y)(x) = \begin{cases} \mathscr{I}(X(x), Y(x)) & X(x) > 0 \\ 0 & X(x) = 0. \end{cases}$$

However, the special treatment of the case $X(x) = 0$ is not necessary to properly define the VPFRS model, hence our simplified definition of $\text{Incl}_{\mathscr{I}}$ in Eq. (9.1). Next, we recall two types of inclusion errors.

Definition 9.1.2. [116] Given a WCP-implicator $\mathscr{I}$, a t-norm $\mathscr{T}$, a non-empty fuzzy set X, a fuzzy set Y in U, and $\alpha \in [0,1]$. The *lower* α*-inclusion error* $\underline{e}_\alpha$ *of* X *in* Y is given by

$$\underline{e}_\alpha(X,Y) = 1 - \frac{|X \cap (\text{Incl}_{\mathscr{I}}(X,Y))_\alpha|}{|X|} \tag{9.3}$$

and the *upper α-inclusion error* $\bar{e}_\alpha$ *of* X *in* Y is given by

$$\bar{e}_\alpha(X,Y) = 1 - \frac{|X \cap ((\mathrm{Incl}_{\mathscr{T}}(X,Y))^{\mathscr{N}_S})_\alpha|}{|X|}, \tag{9.4}$$

where we have used the standard negator $\mathscr{N}_S$. For the empty set $\emptyset$, we define the inclusion errors by

$$\underline{e}_\alpha(\emptyset, Y) = \bar{e}_\alpha(\emptyset, Y) = 0. \tag{9.5}$$

In [116], Mieszkowicz-Rolka and Rolka only defined the inclusion errors for a non-empty fuzzy set X. We extend their definition to include the empty set, in order to allow the use of a general binary fuzzy relation R instead of a fuzzy $\mathscr{T}$-similarity relation in the definition of the VPFRS model:

Definition 9.1.3. [116] Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ a WCP-implicator, $\mathscr{T}$ a t-norm and $0 \leq l < u \leq 1$. The *(u,l)-fuzzy rough approximation operators* $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}})$ are defined as follows: let $X \in \mathscr{F}(U)$ and $x \in U$, then

$$\begin{aligned}
(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}(X))(x) &= \inf_{y \in S_{x,u,X}} \mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(y),\\
(\overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}}(X))(x) &= \sup_{y \in T_{x,l,X}} \mathrm{Incl}_{\mathscr{T}}(R^p(x),X)(y),
\end{aligned}$$

with

$$\begin{aligned}
S_{x,u,X} &= \mathrm{supp}(R^p(x)) \cap \Big((\mathrm{Incl}_{\mathscr{I}}(R^p(x),X))_{\alpha_{x,u,X}}\Big)\\
&= \{y \in U \mid R(y,x) > 0 \text{ and } \mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(y) \geq \alpha_{x,u,X}\},\\
\alpha_{x,u,X} &= \sup\{\alpha \in [0,1] \mid \underline{e}_\alpha(R^p(x),X) \leq 1-u\},\\
T_{x,l,X} &= \mathrm{supp}(R^p(x)) \cap \Big(((\mathrm{Incl}_{\mathscr{T}}(R^p(x),X))^{\mathscr{N}_S})_{\beta_{x,l,X}}\Big)\\
&= \{y \in U \mid R(y,x) > 0 \text{ and } \mathrm{Incl}_{\mathscr{T}}(R^p(x),X)(y) \leq 1-\beta_{x,l,X}\},\\
\beta_{x,l,X} &= \sup\{\alpha \in [0,1] \mid \bar{e}_\alpha(R^p(x),X) \leq l\}.
\end{aligned}$$

Note that for $x \in U$ with $R^p(x) = \emptyset$, it holds that $S_{x,u,X}$ and $T_{x,l,X}$ are empty and $\alpha_{x,u,X} = \beta_{x,l,X} = 1$ for all u and l. In this case, we obtain that $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}(X))(x) = 1$ and $(\overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}}(X))(x) = 0$.

In order to get more insight in this element exclusion process, we show that the approximations of Definition 9.1.3 can be simplified. To do this, we first obtain a simpler expression for $\alpha_{x,u,X}$ and $\beta_{x,l,X}$.

Proposition 9.1.4. Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ a WCP-implicator, $\mathscr{T}$ a t-norm and $0 \leq l < u \leq 1$. For $x \in U$ such that $R^p(x) \neq \emptyset$, it holds for $X \in \mathscr{F}(U)$ that

$$\begin{aligned}
\alpha_{x,u,X} &= \sup\left\{\alpha \in \{\mathscr{I}(R(y,x),X(y)) \mid y \in U\} \mid F_{x,X}(\alpha) \geq u\right\}, \\
&\text{with} \quad F_{x,X}(\alpha) = \frac{|R^p(x) \cap (\mathrm{Incl}_{\mathscr{I}}(R^p(x),X))_\alpha|}{|R^p(x)|}, \\
\beta_{x,l,X} &= \sup\left\{\alpha \in \{1-\mathscr{T}(R(y,x),X(y)) \mid y \in U\} \mid G_{x,X}(\alpha) \geq 1-l\right\}, \\
&\text{with} \quad G_{x,X}(\alpha) = \frac{|R^p(x) \cap ((\mathrm{Incl}_{\mathscr{T}}(R^p(x),X))^{\mathscr{N}_S})_\alpha|}{|R^p(x)|}.
\end{aligned}$$

Proof. Take x in U such that $R^p(x)$ is not empty and $X \in \mathscr{F}(U)$. Since U is finite, we can number and rename its elements such that $U = \{z_1, z_2, \ldots, z_n\}$ and

$$\mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_1) \geq \mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_2) \geq \ldots \geq \mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_n).$$

Define $F_{x,X}: [0,1] \to [0,1]$ such that, for $\alpha \in [0,1]$,

$$F_{x,X}(\alpha) = 1 - \underline{e}_\alpha(R^p(x),X) = \frac{|R^p(x) \cap (\mathrm{Incl}_{\mathscr{I}}(R^p(x),X))_\alpha|}{|R^p(x)|},$$

then $F_{x,X}$ is a decreasing mapping in α. We have the following expression for $\alpha_{x,u,X}$: $\alpha_{x,u,X} = \sup\{\alpha \in [0,1] \mid F_{x,X}(\alpha) \geq u\}$. We prove that we only need to consider the values $\mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(y)$ to compute $\alpha_{x,u,X}$, that is,

$$\alpha_{x,u,X} = \sup\{\alpha \in \{\mathscr{I}(R(y,x),X(y)) \mid y \in U\} \mid F_{x,X}(\alpha) \geq u\}.$$

Take $\alpha^* \in [0,1]$ such that $\alpha^* \notin \{\mathscr{I}(R(y,x),X(y)) \mid y \in U\}$. We prove that this α^* does not influence the supremum. First, assume that α^* is such that $\mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_i) > \alpha^*$ for $i \in \{1,2,\ldots,n\}$, then

$$F_{x,X}(\alpha^*) = F_{x,X}(\mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_i)).$$

Since $\mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_i) > \alpha^*$, we do not need to take α^* into account when we compute $\alpha_{x,u,X}$. On the other hand, if $\alpha^* > \mathrm{Incl}_{\mathscr{I}}(R^p(x),X)(z_1)$, then it holds that

$(\text{Incl}_{\mathcal{I}}(R^p(x), X))_{\alpha^*} = \emptyset$, hence $F_{x,X}(\alpha^*) = 0$. Since we assume $u > 0$, it holds that

$$\alpha^* \notin \{\alpha \in [0,1] \mid F_{x,X}(\alpha) \geq u\}.$$

In both cases we conclude that α^* will not influence the supremum. Hence, we obtain that

$$\begin{aligned} \alpha_{x,u,X} &= \sup\{\alpha \in [0,1] \mid F_{x,X}(\alpha) \geq u\} \\ &= \sup\{\alpha \in \{\mathcal{I}(R(y,x), X(y)) \mid y \in U\} \mid F_{x,X}(\alpha) \geq u\}. \end{aligned}$$

The proof for $\beta_{x,l,X}$ is analogous. □

Given previous proposition, we obtain a simplification of the VPFRS model:

Proposition 9.1.5. Let (U, R) be a fuzzy relation approximation space with U finite, $\mathcal{I}$ a WCP-implicator, $\mathcal{T}$ a t-norm and $0 \leq l < u \leq 1$, then for $X \in \mathcal{F}(U)$ and $x \in U$,

$$\begin{aligned} (\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(X))(x) &= \alpha_{x,u,X}, \\ (\overline{\text{apr}}_{\text{R},\mathcal{T},\text{l}}(X))(x) &= 1 - \beta_{x,l,X}. \end{aligned}$$

Proof. By definition of $S_{x,u,X}$ it holds that

$$(\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(X))(x) = \inf_{y \in S_{x,u,X}} \mathcal{I}(R(y,x), X(y)) \geq \alpha_{x,u,X}.$$

By Proposition 9.1.4, let y^* be the element in U such that supremum is reached, i.e., $\alpha_{x,u,X} = \mathcal{I}(R(y^*,x), X(y^*))$.

- If $R(y^*, x) > 0$, then $y^* \in S_{x,u,X}$ and hence,

$$(\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(X))(x) \leq \mathcal{I}(R(y^*,x), X(y^*)) = \alpha_{x,u,X}.$$

- If $R(y^*, x) = 0$, then it holds that $\alpha_{x,u,X} = \mathcal{I}(0, X(y^*)) = 1$ and thus, $(\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(X))(x) = 1$.

In both cases, we conclude that $(\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(X))(x) = \alpha_{x,u,X}$.

In an analogous way, by definition of $T_{x,l,X}$ we know that

$$(\overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}(X))(x) \leq 1-\beta_{x,l,X}.$$

By Proposition 9.1.4, let z^* be the element in U such that the supremum is reached, i.e., $\beta_{x,l,X} = 1-\mathscr{T}(R(z^*,x),X(z^*))$.

- If $R(z^*,x) > 0$, then $z^* \in T_{x,l,X}$ and hence,

$$(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{u}}(X))(x) \geq \mathscr{T}(R(z^*,x),X(z^*)) = 1-\beta_{x,l,X}.$$

- If $R(z^*,x) = 0$, then it holds that $\beta_{x,l,X} = 1-\mathscr{T}(0,X(z^*)) = 1$ and thus, $(\overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}(X))(x) = 0$.

Hence, we conclude that $(\overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}(X))(x) = 1-\beta_{x,l,X}$. □

The above results give more insight into how the VPFRS model operates. The fuzzy set $\left(R^p(x) \cap (\text{Incl}_{\mathscr{I}}(R^p(x),X))_{\text{Incl}_{\mathscr{I}}(R^p(x),X)(y)}\right)$ can be seen as the fuzzy set of predecessors of x with those elements z excluded, for which

$$\mathscr{I}(R(z,x),X(z)) < \mathscr{I}(R(y,x),X(y)).$$

If the cardinality of this restricted fuzzy set of predecessors is at least a fraction u of that of the entire fuzzy set $R^p(x)$, and there is no smaller such fuzzy set satisfying this condition, then the VPFRS lower approximation equals $\mathscr{I}(R(y,x),X(y))$. An analogous interpretation can be given for the VPFRS upper approximation.

Furthermore, the (u,l)-fuzzy rough approximations operators are very similar to the IC model with respect to the WCP-implicator $\mathscr{I}$ and t-norm $\mathscr{T}$ when the fuzzy neighborhood operator N is considered, with $\text{N}(x)(y) = (R^p(x))(y) = R(y,x)$ for all $x,y \in U$. The only difference is that the infimum and supremum are taken over $S_{x,u,X}$, respectively, $T_{x,l,X}$, instead of the whole universe U. Hence, we obtain that

$$(\underline{\text{apr}}_{\text{R},\mathscr{I}}, \overline{\text{apr}}_{\text{R},\mathscr{T}}) \leq (\underline{\text{apr}}_{\text{R},\mathscr{I},\text{u}}, \overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}).$$

Moreover, the two pairs coincide when $(u,l) = (1,0)$.

Proposition 9.1.6. Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ a WCP-implicator, $\mathscr{T}$ a t-norm and $0 \leq l < u \leq 1$, then

$$(\underline{\text{apr}}_{\text{R},\mathscr{I}}, \overline{\text{apr}}_{\text{R},\mathscr{T}}) \leq (\underline{\text{apr}}_{\text{R},\mathscr{I},\text{u}}, \overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}).$$

Moreover, when $(u,l) = (1,0)$ it holds that

$$(\underline{\text{apr}}_{\text{R},\mathscr{I}}, \overline{\text{apr}}_{\text{R},\mathscr{T}}) = (\underline{\text{apr}}_{\text{R},\mathscr{I},1}, \overline{\text{apr}}_{\text{R},\mathscr{T},0}).$$

Proof. Let $X \in \mathscr{F}(U)$ and $x \in U$. The inequality follows immediately from $S_{x,u,X} \subseteq U$ and $T_{x,l,X} \subseteq U$ for all u and l.

Next, let $u = 1$ and $l = 0$. First, if $R^p(x) = \emptyset$ is empty, then

$$(\underline{\text{apr}}_{\text{R},\mathscr{I},1}(X))(x) = 1 = \inf_{y \in U} \mathscr{I}(0, X(y)) = (\underline{\text{apr}}_{\text{R},\mathscr{I}}(X))(x),$$
$$(\overline{\text{apr}}_{\text{R},\mathscr{T},0}(X))(x) = 0 = \sup_{y \in U} \mathscr{T}(0, X(y)) = (\overline{\text{apr}}_{\text{R},\mathscr{T}}(X))(x),$$

i.e., both pairs coincide. If $R^p(x)$ is not empty, then

$$\alpha_{x,1,X} = \sup\{\alpha \in [0,1] \mid \forall y \in U : R(y,x) > 0 \Rightarrow \mathscr{I}(R(y,x), X(y)) \geq \alpha\}.$$

If $R(y,x) = 0$, then $\mathscr{I}(R(y,x), X(y)) = 1 \geq \alpha$ for any $\alpha \in [0,1]$, so we obtain that $\alpha_{x,1,X} = \inf_{y \in U} \mathscr{I}(R(y,x), X(y)) = (\underline{\text{apr}}_{\text{R},\mathscr{I}}(X))(x)$. By Proposition 9.1.5, we obtain that $(\underline{\text{apr}}_{\text{R},\mathscr{I},1}(X))(x) = (\underline{\text{apr}}_{\text{R},\mathscr{I}}(X))(x)$. For $l = 0$, we derive that

$$\beta_{x,0,X} = \sup\{\alpha \in [0,1] \mid \forall y \in U : R(y,x) > 0 \Rightarrow 1 - \mathscr{T}(R(y,x), X(y)) \geq \alpha\}.$$

If $R(y,x) = 0$, then $1 - \mathscr{T}(R(y,x), X(y)) = 1 \geq \alpha$ for any $\alpha \in [0,1]$, so it holds that

$$\begin{aligned} \beta_{x,0,X} &= \inf_{y \in U} (1 - \mathscr{T}(R(y,x), X(y))) \\ &= 1 - \sup_{y \in U} \mathscr{T}(R(y,x), X(y)) \\ &= 1 - (\overline{\text{apr}}_{\text{R},\mathscr{T}}(X))(x). \end{aligned}$$

By Proposition 9.1.5, we obtain that $(\overline{\text{apr}}_{\text{R},\mathscr{T},0}(X))(x) = (\overline{\text{apr}}_{\text{R},\mathscr{T}}(X))(x)$. □

Since the IC model encapsulates the model of Dubois and Prade [41, 42], we obtain for the Kleene-Dienes implicator, the minimum operator, a fuzzy similarity relation and $(u, l) = (1, 0)$ that the VPFRS model equals the model of Dubois and Prade. Furthermore, when the set X is a crisp set and R is a crisp equivalence relation, the VPFRS model coincides with the VPRS model of Ziarko.

Proposition 9.1.7. Let (U, E) be a Pawlak approximation space with U finite, $\mathscr{I}$ a WCP-implicator, $\mathscr{T}$ a t-norm and $0 \leq l < u \leq 1$, then

$$(\underline{\text{apr}}_{\text{E},\mathscr{I},\text{u}}, \overline{\text{apr}}_{\text{E},\mathscr{T},\text{l}}) = (\underline{\text{apr}}_{\text{E,u}}, \overline{\text{apr}}_{\text{E,l}}).$$

Proof. Let $x \in U$. Since X and E are crisp, $\alpha_{x,u,X}$ and $\beta_{x,l,X}$ are either 1 or 0. First, note that, for $y \in U$,

$$\begin{aligned} y \in [x]_E \cap (\text{Incl}_{\mathscr{I}}([x]_E, X))_1 &\Leftrightarrow E(y,x) = 1 \text{ and } X(y) = 1 \\ &\Leftrightarrow y \in [x]_E \cap X. \end{aligned}$$

For $\alpha_{x,u,X}$ we derive that

$$\begin{aligned} \alpha_{x,u,X} = 1 &\Leftrightarrow \frac{|[x]_E \cap (\text{Incl}_{\mathscr{I}}([x]_E, X))_1|}{|[x]_E|} \geq u \\ &\Leftrightarrow \frac{|[x]_E \cap X|}{|[x]_E|} \geq u \\ &\Leftrightarrow x \in \underline{\text{apr}}_{\text{E,u}}(X). \end{aligned}$$

Moreover, since

$$\begin{aligned} y \in [x]_E \cap ((\text{Incl}_{\mathscr{T}}([x]_E, X))^{\mathscr{N}_S})_1 &\Leftrightarrow E(y,x) = 1 \text{ and } X(y) = 0 \\ &\Leftrightarrow y \in [x]_E \cap X^{\text{c}}, \end{aligned}$$

hence, we obtain for $\beta_{x,l,X}$ that

$$\begin{aligned} \beta_{x,l,X} = 1 &\Leftrightarrow \frac{|[x]_E \cap ((\text{Incl}_{\mathscr{T}}([x]_E, X))^{\mathscr{N}_S})_1|}{|[x]_E|} \geq 1 - l \\ &\Leftrightarrow \frac{|[x]_E \cap X^{\text{c}}|}{|[x]_E|} \geq 1 - l \\ &\Leftrightarrow \frac{|[x]_E \cap X|}{|[x]_E|} \leq l \end{aligned}$$

$$\Leftrightarrow \quad x \notin \overline{\mathrm{apr}}_{\mathrm{E,l}}(X).$$

The proof now follows from Proposition 9.1.5. □

To end, we want to discuss the properties of the VPFRS model. By the above proposition, we only need to discuss the properties (D) for $u = 1 - l$, (SM), (UE) and (CS), since the VPRS model of Ziarko does not satisfy any of the other properties.

Proposition 9.1.8. Let (U, R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ a WCP-implicator, $\mathscr{T}$ a t-norm and $0 \leq l < u \leq 1$.

- The pair $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}})$ satisfies (D) with respect to $\mathscr{N}_S$ if $\mathscr{T}$ is the induced t-norm of $\mathscr{I}$ and $\mathscr{N}_S$ and $u = 1 - l$.
- The pair $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}})$ satisfies (SM).

Proof. • Let $X \in \mathscr{F}(U)$ and $x \in U$. If $R^p(x) = \emptyset$, then

$$(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}(X^{\mathscr{N}_S}))^{\mathscr{N}_S}(x) = \mathscr{N}_S(1) = 0 = (\overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}}(X))(x).$$

If $R^p(x) \neq \emptyset$, let $\alpha \in [0,1]$ and $y \in U$, then

$$\begin{aligned} y \in (\mathrm{Incl}_{\mathscr{I}}(R^p(x), X^{\mathscr{N}_S}))_\alpha &\Leftrightarrow \mathscr{I}(R^p(x)(y), 1 - X(y)) \geq \alpha \\ &\Leftrightarrow 1 - \mathscr{T}(R^p(x)(y), X(y)) \geq \alpha \\ &\Leftrightarrow y \in ((\mathrm{Incl}_{\mathscr{T}}(R^p(x), X))^{\mathscr{N}_S})_\alpha. \end{aligned}$$

Therefore, we obtain that $F_{x,X^{\mathscr{N}_S}}(\alpha) \geq u \Leftrightarrow G_{x,X}(\alpha) \geq u = 1 - l$ and thus,

$$\begin{aligned} \beta_{x,l,X} &= \sup\{\alpha \in \{1 - \mathscr{T}(R(y,x), X(y)) \mid y \in U\} \mid G_{x,X}(\alpha) \geq 1 - l\} \\ &= \sup\{\alpha \in \{\mathscr{I}(R(y,x), 1 - X(y)) \mid y \in U\} \mid F_{x,X^{\mathscr{N}_S}}(\alpha) \geq u\} \\ &= \alpha_{x,u,X^{\mathscr{N}_S}}. \end{aligned}$$

By Proposition 9.1.5, we derive that

$$(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}(X^{\mathscr{N}_S})^{\mathscr{N}_S})(x) = 1 - \alpha_{x,u,X^{\mathscr{N}_S}} = 1 - \beta_{x,l,X} = (\overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}}(X))(x).$$

We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\mathrm{u}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{T},\mathrm{l}})$ satisfies (D) with respect to the standard negator.

- Let X, Y be fuzzy sets in U such that $X \subseteq Y$ and $x \in U$. If $R^p(x) = \emptyset$, then

$$(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{u}}(X))(x) = 1 = (\underline{\text{apr}}_{\text{R},\mathscr{I},\text{u}}(Y))(x),$$
$$(\overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}(X))(x) = 0 = (\overline{\text{apr}}_{\text{R},\mathscr{T},\text{l}}(Y))(x).$$

Now, assume $R^p(x) \neq \emptyset$. Due to Proposition 9.1.5, we have to prove that $\alpha_{x,u,X} \leq \alpha_{x,u,Y}$ and $\beta_{x,l,X} \geq \beta_{x,l,Y}$. First, we reorder the elements of U such that $U = \{z_1, z_2, \ldots, z_n\}$ and

$$\mathscr{I}(R(z_1, x), X(z_1)) \geq \mathscr{I}(R(z_2, x), X(z_2)) \geq \ldots \geq \mathscr{I}(R(z_n, x), X(z_n)).$$

By Proposition 9.1.4, there exists an $m \in \{1, 2, \ldots, n\}$ such that

$$\alpha_{x,u,X} = \mathscr{I}(R(z_m, x), X(z_m)).$$

Now, for every $i \in \{1, 2, \ldots, n\}$ it holds that

$$\mathscr{I}(R(z_i, x), Y(z_i)) \geq \mathscr{I}(R(z_i, x), X(z_i))$$

and thus for all $i \in \{1, 2, \ldots, m\}$,

$$\mathscr{I}(R(z_i, x), Y(z_i)) \geq \alpha_{x,u,X}.$$

By definition of $\alpha_{x,u,X}$, it holds that

$$\sum_{\substack{z \in U: \\ \mathscr{I}(R(z,x),X(z)) \geq \alpha_{x,u,X}}} R(z, x) \geq u \cdot \sum_{z \in U} R(z, x)$$

and thus it also holds that

$$\sum_{\substack{z \in U: \\ \mathscr{I}(R(z,x),Y(z)) \geq \alpha_{x,u,X}}} R(z, x) \geq u \cdot \sum_{z \in U} R(z, x).$$

In other words,

$$\alpha_{x,u,X} \in \left\{\alpha \in [0, 1] \mid \underline{e}_\alpha(R^p(x), Y) \leq 1 - u\right\}$$

Hence, $\alpha_{x,u,X} \leq \alpha_{x,u,Y}$. In an analogous way, we obtain that

$$\beta_{x,l,Y} \in \{\alpha \in [0, 1] \mid \bar{e}_\alpha(R^p(x), X) \leq l\},$$

and thus, $\beta_{x,l,Y} \leq \beta_{x,l,X}$.

□

Note that the duality property is limited to the standard negator, since $\mathcal{N}_S$ is used in the definition of $T_{x,l,X}$ and $\beta_{x,l,X}$ for $X \in \mathcal{F}(U)$ and $x \in U$. The VPFRS model does not satisfy (UE) and (CS) as illustrated in the next example.

Example 9.1.9. Let $U = \{x, y, z\}$, $\mathcal{T}$ the Łukasiewicz t-norm and $\mathcal{I}$ the Łukasiewicz implicator and let R be the fuzzy $\mathcal{T}$-similarity relation defined by $R(x, y) = 0.7$, $R(x,z) = 0.8$ and $R(y,z) = 0.8$. Let $l = 0.4$ and $u = 0.6$, then $(\underline{\text{apr}}_{\text{R},\mathcal{I},\text{u}}(\emptyset))(y) = 0.2$ and $(\overline{\text{apr}}_{\text{R},\mathcal{T},\text{l}}(U))(y) = 0.8$. We conclude that the properties (UE) and (CS) are not satisfied.

We continue with Cornelis et al.'s vaguely quantified fuzzy rough set model.

9.1.2 Vaguely quantified fuzzy rough set model

In 2007, Cornelis et al. [16] introduced the Vaguely Quantified Fuzzy Rough Set (VQFRS) model. In contrast to the other fuzzy rough set approaches, they did not make use of implicators and conjunctors, but they worked with fuzzy quantifiers to extend Ziarko's VPRS model.

Definition 9.1.10. [16] A *regularly increasing fuzzy quantifier* is an increasing mapping $Q\colon [0,1] \to [0,1]$ which satisfies $Q(0) = 0$ and $Q(1) = 1$.

The VQFRS model is then defined based on two such regularly increasing fuzzy quantifiers.

Definition 9.1.11. [16] Let (U,R) be a fuzzy relation approximation space with U finite and let (Q_u, Q_l) be a pair of regularly increasing fuzzy quantifiers. The pair of (Q_u, Q_l)-*vaguely quantified fuzzy rough approximation operators* $(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}, \overline{\text{apr}}_{\text{R},\text{Q}_\text{l}})$ is defined as follows: let $X \in \mathcal{F}(U)$ and $x \in U$, then

$$
\begin{aligned}
(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}(X))(x) &= \begin{cases} Q_u\left(\frac{|R^p(x)\cap X|}{|R^p(x)|}\right) & R^p(x) \neq \emptyset \\ 1 & R^p(x) = \emptyset, \end{cases} \\
(\overline{\text{apr}}_{\text{R},\text{Q}_\text{l}}(X))(x) &= \begin{cases} Q_l\left(\frac{|R^p(x)\cap X|}{|R^p(x)|}\right) & R^p(x) \neq \emptyset \\ 1 & R^p(x) = \emptyset. \end{cases}
\end{aligned}
$$

In [16], the interpretation of this model is as follows: x belongs to the lower approximation of X if 'most' of the elements related to x belong to X and x belongs to the upper approximation of X if 'at least some' elements related to x belong to X. The linguistic quantifiers 'most' and 'at least some' generalize the crisp $\forall$- and $\exists$-quantifiers, and are modeled by means of the fuzzy quantifiers Q_u and Q_l, respectively. In [16], as a specific example, the authors put $Q_u = Q_{(0.2,1)}$ and $Q_l = Q_{(0.1,0.6)}$, where $Q_{(a,b)}$ with $a, b \in [0,1]$ is defined by

$$\forall c \in [0,1]\colon Q_{(a,b)}(c) = \begin{cases} 0 & c \leq a \\ \frac{2(c-a)^2}{(b-a)^2} & a \leq c \leq \frac{a+b}{2} \\ 1 - \frac{2(c-b)^2}{(b-a)^2} & \frac{a+b}{2} \leq c \leq b \\ 1 & b \leq c. \end{cases}$$

As pointed out in [16], the VQFRS model has Pawlak's model and Ziarko's VPRS model as specific cases when X is a crisp set and R is a crisp equivalence relation. In the former case, let $Q_u = Q_\forall$ and $Q_l = Q_\exists$, where

$$\forall c \in [0,1]\colon Q_\forall(c) = \begin{cases} 0 & c < 1 \\ 1 & c = 1, \end{cases}$$

$$\forall c \in [0,1]\colon Q_\exists(c) = \begin{cases} 0 & c = 0 \\ 1 & c > 0. \end{cases}$$

In the case of the VPRS model, let $Q_u = Q_{\geq u}$ and $Q_l = Q_{>l}$, where

$$\forall c \in [0,1]\colon Q_{\geq u}(c) = \begin{cases} 0 & c < u \\ 1 & c \geq u, \end{cases}$$

$$\forall c \in [0,1]\colon Q_{>l}(c) = \begin{cases} 0 & c \leq l \\ 1 & c > l. \end{cases}$$

There is no connection between the VQFRS model and the IC model, i.e., in general, we cannot find fuzzy quantifiers Q_u and Q_l such that

$$(\underline{\mathrm{apr}}_{\mathrm{R},\mathrm{Q}_\mathrm{u}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathrm{Q}_\mathrm{l}}) = (\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I}}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{C}})$$

for a certain implicator $\mathscr{I}$ and conjunctor $\mathscr{C}$.

To end, we discuss the properties of the VQFRS model. Since the VPRS model of Ziarko is a special case of the VQFRS model, we only discuss (D), (SM), (UE) and (CS). It can be verified that the property (SM) is satisfied and the property (UE) is satisfied if R is inverse serial.

Proposition 9.1.12. Let (U,R) be a fuzzy relation approximation space with U finite and let (Q_u,Q_l) be a pair of regularly increasing fuzzy quantifiers.

- The pair $(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}, \overline{\text{apr}}_{\text{R},\text{Q}_\text{l}})$ satisfies (SM).
- The pair $(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}, \overline{\text{apr}}_{\text{R},\text{Q}_\text{l}})$ satisfies (UE) if R is a inverse serial[6] fuzzy relation.

Proof. The property (SM) follows from the monotonicity of regularly increasing fuzzy quantifiers. Next, let R be a fuzzy serial relation, then for all $x \in U$ it holds that

$$Q_u\left(\frac{|R^p(x)\cap\emptyset|}{|R^p(x)|}\right) = Q_l\left(\frac{|R^p(x)\cap\emptyset|}{|R^p(x)|}\right) = 0$$

and

$$Q_u\left(\frac{|R^p(x)\cap U|}{|R^p(x)|}\right) = Q_l\left(\frac{|R^p(x)\cap U|}{|R^p(x)|}\right) = 1.$$

□

It is very easy to see that the condition on R is necessary for the property (UE): let $R^p(x) = \emptyset$ for $x \in U$, then $(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}(\emptyset))(x) = 1$. The properties (D) and (CS) are not satisfied, as illustrated in the next example:

Example 9.1.13. Let $U = \{x,y,z\}$ and R the fuzzy set $U \times U$. Let $\alpha = 0.3$, $Q_u = Q_{(0.2,1)}$ and $Q_l = Q_{(0.1,0.6)}$, then $(\underline{\text{apr}}_{\text{R},\text{Q}_\text{u}}(\hat{\alpha}))(x) = Q_u(0.3) = \frac{1}{32}$. Thus, (CS) is not satisfied. Moreover, consider the standard negator $\mathscr{N}_S$. We derive that $(\overline{\text{apr}}_{\text{R},\text{Q}_\text{l}}(\hat{\alpha}^{\mathscr{N}_S}))^{\mathscr{N}_S}(x) = 1 - Q_l(0.7) = 0$. Hence, the duality property is not satisfied.

Next, we discuss Hu et al.'s soft fuzzy rough set model.

[6]If U is infinite, the fuzzy relation R should be strongly inverse serial.

9.1.3 Soft fuzzy rough set model

Inspired by soft margin support vector machines [20], Hu et al. [69] proposed in 2010 the Soft Fuzzy Rough Set (SFRS) model as a new robust fuzzy rough set model. An important ingredient of the model is the so-called *soft distance* between an element $x \in U$ and a crisp set $X \subseteq U$, defined as follows in [70]:

$$\mathrm{SD}(x,X) = \arg_{d(x,y)} \sup\{d(x,y) - \delta \cdot m_{x,y} \mid y \in X\},$$

where d is a distance funtion, $\delta > 0$ is a penalty factor and

$$m_{x,y} = |\{z \in U \mid d(x,z) < d(x,y)\}|.$$

However, we may encounter a problem with this definition due to the use of the function $\arg_{d(x,y)}$: when the value of the supremum is reached for different values of y, it is not clear which y should generate the soft distance. The following example illustrates this.

Example 9.1.14. Let $U = \{x, y_1, y_2, y_3\}$, $X = \{y_1, y_2, y_3\}$, $\delta = 0.1$ and

$$d(x,x) = 0, d(x,y_1) = 0.2, d(x,y_2) = 0.3, d(x,y_3) = 0.4.$$

Because $d(x,y_1) - \delta \cdot m_{x,y_1} = d(x,y_2) - \delta \cdot m_{x,y_2} = d(x,y_3) - \delta \cdot m_{x,y_3} = 0.1$, $\mathrm{SD}(x,X)$ could be either 0.2, 0.3 or 0.4.

Based on the soft distance, the authors defined the SFRS model with a distance function d determined by $d(x,y) = 1 - R(y,x)$ for all $x,y \in U$ and R a fuzzy relation. However, since the use of the arg function leads to ambiguity as illustrated above, we introduce a slightly adapted definition of the model.

Definition 9.1.15. Let (U,R) be a fuzzy relation approximation space with U, $\mathscr{I}$ an implicator, $\mathscr{C}$ a conjunctor and $\delta > 0$ a penalty factor. The *soft fuzzy rough approximation operators* $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\delta}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{C},\delta})$ are defined by, for $X \in \mathscr{F}(U)$, $x \in U$,

$$\begin{aligned}
(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\delta}(X))(x) &= \mathscr{N}_{\mathscr{I}}\left(\inf_{y \in \Omega_x} R(y,x)\right), \\
(\overline{\mathrm{apr}}_{\mathrm{R},\mathscr{C},\delta}(X))(x) &= \inf_{y \in \Pi_x} R(y,x),
\end{aligned}$$

with

$$\begin{aligned}
\Omega_x &= \big\{z \in U \mid X(z) \leq \mu_x \text{ and } (\forall z' \in U)(X(z') \leq \mu_x \\
&\qquad \Rightarrow R(z',x) + \delta \cdot m_{x,z'} \geq R(z,x) + \delta \cdot m_{x,z})\big\},\\
\mu_x &= \sup\{X(z) \mid z \in U \text{ and } \mathscr{I}(R(z,x),X(z)) = \sigma_x\},\\
\sigma_x &= \inf_{z\in U} \mathscr{I}(R(z,x),X(z)),\\
m_{x,z} &= |\{w \in U \mid X(w) \leq \mu_x \text{ and } R(w,x) > R(z,x)\}|
\end{aligned}$$

and

$$\begin{aligned}
\Pi_x &= \big\{z \in U \mid X(z) \geq \nu_x \text{ and } (\forall z' \in U)(X(z') \geq \nu_x \\
&\qquad \Rightarrow R(z',x) + \delta \cdot n_{x,z'} \geq R(z,x) + \delta \cdot n_{x,z})\big\},\\
\nu_x &= \inf\{X(z) \mid z \in U \text{ and } \mathscr{C}(R(z,x),X(z)) = \tau_x\},\\
\tau_x &= \sup_{z\in U} \mathscr{C}(R(z,x),X(z)),\\
n_{x,z} &= |\{w \in U \mid X(w) \geq \nu_x \text{ and } R(w,x) > R(z,x)\}|.
\end{aligned}$$

We briefly explain the intuition behind the lower approximation, the explanation for the upper approximation is analogous. First, we identify those elements z in U for which the value of the infimum considered in the IC lower approximation, $\sigma_x = (\underline{\text{apr}}_{R,\mathscr{I}}(X))(x)$, is reached. As there may be several of them, we consider the one that has the highest membership to X, we denote this membership degree by μ_x. To obtain Ω_x, we look for z in U such that $X(z) \leq \mu_x$ and $R(z,x) + \delta \cdot m_{x,z}$ is minimal, where $m_{x,z}$ counts the number of elements w in U such that $X(w) \leq \mu_x$ and $R(w,x) > R(z,x)$. Among all the $y \in \Omega_x$, we choose the one that has the smallest value of $R(y,x)$. Finally, to compute the lower approximation in x, we take the negation of that value $R(y,x)$.

By tuning the penalty factor δ, we may allow for more or less noise tolerance. If δ is sufficiently large, the result will be determined by the largest value of $R(z,x)$ (or equivalently, the smallest value of $m_{x,z}$) among the considered z, since other elements will not satisfy the minimality condition. For small values of δ, more noise tolerance is allowed. In this case, the largest values of $R(z,x)$ are overlooked, and consequently the membership degree to the lower approximation gets larger.

To see the connection with the soft distance, note that taking z such that $R(z,x)+\delta \cdot m_{x,z}$ is minimal is equivalent by taking z such that $1-R(z,x)-\delta \cdot m_{x,z}$ is maximal. In other words, we consider the soft distance $\mathrm{SD}(z, R^p(x))$.

In general, there is no relationship between the SFRS model and the IC model. To end, we discuss the properties of the SFRS model. The fuzzy approximation operators satisfy the duality with respect to the considered involutive negator.

Proposition 9.1.16. Let (U,R) be a fuzzy relation approximation space with U, $\mathscr{I}$ an implicator and $\delta > 0$ a penalty factor. If the induced negator $\mathscr{N}_{\mathscr{I}}$ is involutive and $\mathscr{C}$ is the induced conjunctor of $\mathscr{I}$ and $\mathscr{N}_{\mathscr{I}}$, then the pair $(\underline{\mathrm{apr}}_{\mathrm{R},\mathscr{I},\delta}, \overline{\mathrm{apr}}_{\mathrm{R},\mathscr{C},\delta})$ satisfies (D) with respect to $\mathscr{N}_{\mathscr{I}}$.

Proof. Let $X \in \mathscr{F}(U)$ and $x \in U$. Since $\mathscr{N}_{\mathscr{I}}$ is involutive, it is continuous. For the lower approximation of $X^{\mathscr{N}_{\mathscr{I}}}$ in x and the upper approximation of X in x we obtain that:

$$
\begin{aligned}
\sigma_x &= \inf_{z \in U} \mathscr{I}(R(z,x), \mathscr{N}_{\mathscr{I}}(X(z))) \\
&= \inf_{z \in U} \mathscr{N}_{\mathscr{I}}(\mathscr{C}(R(z,x), X(z))) \\
&= \mathscr{N}_{\mathscr{I}}\left(\sup_{z \in U} \mathscr{C}(R(z,x), X(z))\right) \\
&= \mathscr{N}_{\mathscr{I}}(\tau_x), \\
\mu_x &= \sup\{\mathscr{N}_{\mathscr{I}}(X(z)) \mid z \in U \text{ and } \mathscr{I}(R(z,x), \mathscr{N}_{\mathscr{I}}(X(z))) = \sigma_x\} \\
&= \mathscr{N}_{\mathscr{I}}\left(\inf\{X(z) \mid z \in U \text{ and } \mathscr{N}_{\mathscr{I}}(\mathscr{C}(R(z,x), X(z))) = \mathscr{N}_{\mathscr{I}}(\tau_x)\}\right) \\
&= \mathscr{N}_{\mathscr{I}}\left(\inf\{X(z) \mid z \in U \text{ and } \mathscr{C}(R(z,x), X(z)) = \tau_x\}\right) \\
&= \mathscr{N}_{\mathscr{I}}(\nu_x).
\end{aligned}
$$

Furthermore, we have for every $z \in U$ that

$$
\begin{aligned}
m_{x,z} &= |\{w \in U \mid \mathscr{N}_{\mathscr{I}}(X(w)) \leq \mu_x \wedge R(w,x) > R(z,x)\}| \\
&= |\{w \in U \mid X(w) \geq \mathscr{N}_{\mathscr{I}}(\mu_x) \wedge R(w,x) > R(z,x)\}| \\
&= |\{w \in U \mid X(w) \geq \nu_x \wedge R(w,x) > R(z,x)\}| \\
&= n_{x,z}.
\end{aligned}
$$

Hence,

$$\begin{aligned}\Omega_x &= \big\{z \in U \mid \mathcal{N}_{\mathcal{I}}(X(z)) \leq \mu_x \text{ and } (\forall z' \in U)(\mathcal{N}_{\mathcal{I}}(X(z')) \leq \mu_x \\ &\quad \Rightarrow R(z',x) + \delta \cdot m_{x,z'} \geq R(z,x) + \delta \cdot m_{x,z})\big\}, \\ &= \big\{z \in U \mid X(z) \geq \nu_x \text{ and } (\forall z' \in U)(X(z') \geq \nu_x \\ &\quad \Rightarrow R(z',x) + \delta \cdot n_{x,z'} \geq R(z,x) + \delta \cdot n_{x,z})\big\}, \\ &= \Pi_x.\end{aligned}$$

We conclude that

$$\begin{aligned}(\underline{\text{apr}}_{R,\mathcal{I},\delta}(X^{\mathcal{N}_{\mathcal{I}}}))^{\mathcal{N}_{\mathcal{I}}}(x) &= \inf_{y \in \Omega_x} R(y,x) \\ &= \inf_{y \in \Pi_x} R(y,x) \\ &= (\overline{\text{apr}}_{R,\mathcal{C},\delta}(X))(x).\end{aligned}$$

□

In particular, Proposition 9.1.16 holds if the pair $(\mathcal{I},\mathcal{T})$ consists of an S-implicator based on a t-conorm $\mathcal{S}$ and $\mathcal{N}_{\mathcal{I}}$ with $\mathcal{S}$ the $\mathcal{N}_{\mathcal{I}}$-dual of $\mathcal{T}$, or if it consists of an IMTL-t-norm and its R-implicator.

Other properties do not hold for the pair $(\underline{\text{apr}}_{R,\mathcal{I},\delta}, \overline{\text{apr}}_{R,\mathcal{C},\delta})$, although in [69] the authors claimed that (IU) is satisfied for the Kleene-Dienes implicator, the minimum operator and a fuzzy similarity relation. However, below we show that this claim is false.

Example 9.1.17. Let $U = \{x,y,z\}$, $\mathcal{I}$ the Kleene-Dienes implicator, $\mathcal{T}$ the minimum operator and $\delta = 0.1$. Note that the induced negator of $\mathcal{I}$ is the standard negator. Let R be a fuzzy similarity relation with $R(x,y) = 0.8$, $R(x,z) = R(y,z) = 0.4$. Let $X = 0.4/x + 0.5/y + 0.2/z$ and $Y = 0.8/x + 0.5/y + 0.8/z$, then $X \subseteq Y$, and thus $X \cap Y = X$. We obtain that

(a) $(\underline{\text{apr}}_{R,\mathcal{I},\delta}(X))(x) = (\underline{\text{apr}}_{R,\mathcal{I},\delta}(X \cap Y))(x) = 1 - R(z,x) = 0.6$,

(b) $(\underline{\text{apr}}_{R,\mathcal{I},\delta}(Y))(x) = 1 - R(y,x) = 0.2$, hence, $\underline{\text{apr}}_{R,\mathscr{I},\delta}(X \cap Y) \neq \underline{\text{apr}}_{R,\mathscr{I},\delta}(X) \cap \underline{\text{apr}}_{R,\mathscr{I},\delta}(Y)$, thus, the property (IU) is not satisfied. Moreover, the properties (INC) and (SM)[7] are not satisfied, since

(a) $(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X))(x) = 0.6 > 0.4 = X(x)$,

(b) $(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X))(x) = 0.6 > 0.2 = (\underline{\text{apr}}_{R,\mathscr{I},\delta}(Y))(x)$.

In addition, we have that

(a) $\underline{\text{apr}}_{R,\mathscr{I},\delta}(X) = \hat{0.6} = \underline{\text{apr}}_{R,\mathscr{I},\delta}(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X))$,

(b) $\overline{\text{apr}}_{R,\mathscr{C},\delta}(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X)) = 0.8/x + 0.8/y + 0.4/z$.

Hence, as $(\overline{\text{apr}}_{R,\mathscr{C},\delta}(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X)))(x) = 0.8 > 0.6 = (\underline{\text{apr}}_{R,\mathscr{I},\delta}(X))(x)$, the properties (CS) and (LU) do not hold. The property (A) does not hold either, since $\underline{\text{apr}}_{R,\mathscr{I},\delta}(\hat{0.6}) \subseteq \hat{0.6}$, but $\hat{0.6} \nsubseteq \overline{\text{apr}}_{R,\mathscr{C},\delta}(\hat{0.6})$. The property (UE) is not satisfied, since $(\underline{\text{apr}}_{R,\mathscr{I},\delta}(\emptyset))(x) = 0.6$. Furthermore, since

(a) $\underline{\text{apr}}_{R,\mathscr{I},\delta}(Y) = 0.2/x + 0.6/y + 0.6/z$,

(b) $\underline{\text{apr}}_{R,\mathscr{I},\delta}(\underline{\text{apr}}_{R,\mathscr{I},\delta}(Y)) = 0.2/x + 0.2/y + 0.6/z$,

the property (ID) is not satisfied for a left-continuous t-norm $\mathscr{T}$ and an S-implicator based on the $\mathscr{N}$-dual of $\mathscr{T}$ and $\mathscr{N}$.

The property (ID) does not hold for a left-continuous t-norm and its R-implicator either. Consider the minimum operator and the Gödel implicator. Note that the induced negator is the Gödel negator. Let R be the fuzzy similarity relation with $R(x,y) = 0.1$ and $R(x,z) = R(y,z) = 0$. Let $\delta = 0.2$, then for the fuzzy set $X = 0.7/x + 0.2/y + 0.1/z$, it holds that $\underline{\text{apr}}_{R,\mathscr{I},\delta}(X) = 0/x + 1/y + 0/z$. However,

$$(\underline{\text{apr}}_{R,\mathscr{I},\delta}(\underline{\text{apr}}_{R,\mathscr{I},\delta}(X)))(y) = 0,$$

thus, the (ID) property is not satisfied.

Finally, let $U = \{x, y, z\}$ and let R_1 and R_2 be fuzzy similarity relations with $R_1(x,y) = 0.4$, $R_2(x,y) = 0.9$ and $R_1(x,z) = R_1(y,z) = R_2(x,z) = R_2(y,z) = 0.3$, then $R_1 \subseteq R_2$. Consider the Łukasiewicz implicator and $\delta = 0.2$, then we obtain for $X = 0.6/x + 0.6/y + 0.2/z$,

[7] In [35], it was shown that (SM) is not satisfied for crisp subsets of U.

(a) $(\underline{\text{apr}}_{R_1,\mathscr{I},\delta}(X))(x) = 1 - R_1(y,x) = 0.6$,

(b) $(\underline{\text{apr}}_{R_2,\mathscr{I},\delta}(X))(x) = 1 - R_2(z,x) = 0.7$,

thus $(\underline{\text{apr}}_{R_1,\mathscr{I},\delta}(X))(x) < (\underline{\text{apr}}_{R_2,\mathscr{I},\delta}(X))(x)$. We conclude that (RM) is not satisfied.

The final noise-tolerant model based on frequency we discuss is Yao et al.'s variable precision fuzzy rough set model based on fuzzy granules.

9.1.4 Variable precision fuzzy rough set model based on fuzzy granules

The last model we discuss based on frequency was proposed in 2014 by Yao et al. [178] and is a granule-based fuzzy rough set model instead of an element-based one. In this approach, called FG model henceforth, the authors work with a left-continuous t-norm $\mathscr{T}$, its $\mathscr{N}_S$-dual t-conorm $\mathscr{S}$ and a fuzzy $\mathscr{T}$-similarity relation R.

The relationship between fuzzy granules and fuzzy rough set theory was discussed for the first time by Chen et al. in [13]. In traditional set theory, a set is determined by the elements it contains. In fuzzy set theory, a fuzzy point plays a similar role: a non-empty fuzzy set is the union of certain fuzzy points. Fuzzy granules can then be constructed around these fuzzy points.

Definition 9.1.18. [13] Let $x \in U$ and $\lambda \in (0,1]$. A *fuzzy point* is a fuzzy set x_λ in U defined by

$$\forall y \in U\colon x_\lambda(y) = \begin{cases} \lambda & y = x \\ 0 & y \neq x. \end{cases}$$

Note that the fuzzy point x_1 was previous denoted by the fuzzy set 1_x.

Definition 9.1.19. [13] Let (U,R) be a fuzzy relation approximation space with U finite, $x \in U$ and $\lambda \in [0,1]$. Let $\mathscr{T}$ be a left-continuous t-norm and $\mathscr{S}$ its $\mathscr{N}_S$-dual t-conorm. The *fuzzy information granules* $[x_\lambda]_R^{\mathscr{T}}$ and $[x_\lambda]_R^{\mathscr{S}}$ are defined by:

$$\begin{aligned} \forall y \in U\colon [x_\lambda]_R^{\mathscr{T}}(y) &= \mathscr{T}(R(y,x),\lambda), \\ \forall y \in U\colon [x_\lambda]_R^{\mathscr{S}}(y) &= \mathscr{S}(1-R(y,x),1-\lambda) \end{aligned}$$

$$= \quad 1 - \mathscr{T}(R(y,x),\lambda).$$

Note that $[x_\lambda]_R^{\mathscr{T}}$ and $[x_\lambda]_R^{\mathscr{S}}$ can be seen as fuzzy neighborhoods of the object $x \in U$. However, the membership degree in x depends on the value of λ.

Given the fuzzy information granules, we have the following characterization for the IC approximation operators $\underline{\text{apr}}_{\text{R},\mathscr{I}}$ en $\overline{\text{apr}}_{\text{R},\mathscr{T}}$.

Proposition 9.1.20. [178] Let (U,R) be a fuzzy relation approximation space with U finite and R a fuzzy $\mathscr{T}$-similarity relation for $\mathscr{T}$ a left-continuous t-norm and $\mathscr{I}$ its R-implicator, then for $X \in \mathscr{F}(U)$ we have

$$\underline{\text{apr}}_{\text{R},\mathscr{I}}(X) = \bigcup \left\{ [x_\lambda]_R^{\mathscr{T}} \mid x \in U, \lambda \in [0,1], [x_\lambda]_R^{\mathscr{T}} \subseteq X \right\}.$$

Moreover, if $\mathscr{T}$ is an IMTL-t-norm and $\mathscr{I}$ is its R-implicator such that the involutive negator of $\mathscr{I}$ is the standard negator, it holds that

$$\overline{\text{apr}}_{\text{R},\mathscr{T}}(X) = \bigcap \left\{ [x_\lambda]_R^{\mathscr{S}} \mid x \in U, \lambda \in [0,1], X \subseteq [x_\lambda]_R^{\mathscr{S}} \right\},$$

with $\mathscr{S}$ the $\mathscr{N}_{\mathscr{I}}$-dual of $\mathscr{T}$.

Note that the result in the previous proposition can be extended to any involutive negator $\mathscr{N}$ if the fuzzy granules in Definition 9.1.19 would be defined with respect to this negator $\mathscr{N}$.

Now, instead of considering the fuzzy granules $[x_\lambda]_R^{\mathscr{T}}$ and $[x_\lambda]_R^{\mathscr{S}}$ such that $[x_\lambda]_R^{\mathscr{T}}(y) \leq X(y)$, respectively $X(y) \leq [x_\lambda]_R^{\mathscr{S}}(y)$, holds for all $y \in U$, Yao et al. considered the fuzzy granules such that for 'many' elements y these inequalities hold. This is controlled by a parameter $\gamma \leq 1$ which is recommended to be chosen very close to 1. In [178] the relation R is a $\mathscr{T}$-similarity relation, we consider the model for arbitrary binary fuzzy relations, as was also done simultaneously in a paper by Wang and Hu [164].

Definition 9.1.21. [164, 178] Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{T}$ a left-continuous t-norm, $\mathscr{S}$ the $\mathscr{N}_S$-dual t-conorm of $\mathscr{T}$ and

$\gamma \in [0,1]$. The *fuzzy rough approximation operators based on fuzzy granules* $(\underline{\text{apr}}_{R,\mathcal{T},\gamma}, \overline{\text{apr}}_{R,\mathcal{S},\gamma})$ are defined by, for $X \in \mathcal{F}(U)$ and $x \in U$,

$$\begin{aligned}
&(\underline{\text{apr}}_{R,\mathcal{T},\gamma}(X))(x)\\
= \; &\sup\left\{[z_\lambda]_R^{\mathcal{T}}(x) \mid z \in U, \lambda \in [0,1], \frac{|\{y \in U \mid [z_\lambda]_R^{\mathcal{T}}(y) \leq X(y)\}|}{|U|} \geq \gamma\right\},\\
&(\overline{\text{apr}}_{R,\mathcal{S},\gamma}(X))(x)\\
= \; &\inf\left\{[z_\lambda]_R^{\mathcal{S}}(x) \mid z \in U, \lambda \in [0,1], \frac{|\{y \in U \mid X(y) \leq [z_\lambda]_R^{\mathcal{S}}(y)\}|}{|U|} \geq \gamma\right\}.
\end{aligned}$$

If $\gamma = 1$, R is a fuzzy $\mathcal{T}$-similarity relation, $\mathcal{T}$ is an IMTL-t-norm and $\mathcal{I}$ its R-implicator such that $\mathcal{N}_{\mathcal{I}} = \mathcal{N}_S$ this model coincides with the IC model. For general $\gamma \in [0,1)$, it holds that $(\underline{\text{apr}}_{R,\mathcal{I}}, \overline{\text{apr}}_{R,\mathcal{T}}) \leq (\underline{\text{apr}}_{R,\mathcal{T},\gamma}, \overline{\text{apr}}_{R,\mathcal{S},\gamma})$.

In [178], another characterization of the FG model was proposed. It is proven that for a fuzzy $\mathcal{T}$-similarity relation R, the supremum, respectively the infimum, is reached in x, when computing the lower, respectively the upper, approximation in $x \in U$.

Proposition 9.1.22. [178] Let (U,R) be a fuzzy relation approximation space with U finite, $\mathcal{T}$ a left-continuous t-norm, $\mathcal{S}$ the $\mathcal{N}_S$-dual t-conorm of $\mathcal{T}$ and $\gamma \in [0,1]$. It holds, for $X \in \mathcal{F}(U)$ and $x \in U$, that

$$\begin{aligned}
&(\underline{\text{apr}}_{R,\mathcal{T},\gamma}(X))(x)\\
= \; &\sup\left\{[x_\lambda]_R^{\mathcal{T}}(x) \mid \lambda \in [0,1], \frac{|\{y \in U \mid [x_\lambda]_R^{\mathcal{T}}(y) \leq X(y)\}|}{|U|} \geq \gamma\right\},\\
&(\overline{\text{apr}}_{R,\mathcal{S},\gamma}(X))(x)\\
= \; &\inf\left\{[x_\lambda]_R^{\mathcal{S}}(x) \mid \lambda \in [0,1], \frac{|\{y \in U \mid X(y) \leq [x_\lambda]_R^{\mathcal{S}}(y)\}|}{|U|} \geq \gamma\right\}.
\end{aligned}$$

Note that it suffices that R is reflexive and $\mathcal{T}$-transitive for Proposition 9.1.22 to hold. Next, we discuss the properties of the FG model.

Proposition 9.1.23. Let (U,R) be a fuzzy relation approximation space with U finite, $\mathcal{T}$ a left-continuous t-norm, $\mathcal{S}$ the $\mathcal{N}_S$-dual t-conorm of $\mathcal{T}$ and $\gamma \in [0,1]$.

- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{T},\gamma}, \overline{\text{apr}}_{\text{R},\mathcal{S},\gamma})$ satisfies (D) with respect to the standard negator.
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{T},\gamma}, \overline{\text{apr}}_{\text{R},\mathcal{S},\gamma})$ satisfies (SM).
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{T},\gamma}, \overline{\text{apr}}_{\text{R},\mathcal{S},\gamma})$ satisfies (RM), when the considered fuzzy relations are reflexive and $\mathcal{T}$-transitive.

Proof. Properties (D) and (SM) were proven in [178]. Let $R_1 \subseteq R_2$ be reflexive and $\mathcal{T}$-transitive relations and let $X \in \mathcal{F}(U)$. We introduce the notation λ_x^i for $x \in U$ and $i = 1, 2$:

$$\lambda_x^i = \sup\left\{\lambda \in [0,1] \mid \frac{|\{y \in U \mid [x_\lambda]_{R_i}^{\mathcal{T}}(y) \leq X(y)\}|}{|U|} \geq \gamma\right\}.$$

Due to Proposition 9.1.22 and the left-continuity of $\mathcal{T}$, we have for $x \in U$ and $i = 1, 2$:

$$\begin{aligned}
& (\underline{\text{apr}}_{R_i,\mathcal{T},\gamma}(X))(x) \\
= \ & \sup\left\{[x_\lambda]_{R_i}^{\mathcal{T}}(x) \mid \lambda \in [0,1], \frac{|\{y \in U \mid [x_\lambda]_{R_i}^{\mathcal{T}}(y) \leq X(y)\}|}{|U|} \geq \gamma\right\} \\
= \ & [x_{\lambda_x^i}]_{R_i}^{\mathcal{T}}(x) \\
= \ & \mathcal{T}(R_i(x,x), \lambda_x^i) \\
= \ & \lambda_x^i.
\end{aligned}$$

Now, since $R_1 \subseteq R_2$, it holds for all $x \in U$ and $\lambda \in [0,1]$ that

$$[x_\lambda]_{R_1}^{\mathcal{T}} \subseteq [x_\lambda]_{R_2}^{\mathcal{T}}.$$

Hence $\lambda_x^2 \leq \lambda_x^1$ for all $x \in U$ and thus, $\underline{\text{apr}}_{R_2,\mathcal{T},\gamma}(X) \subseteq \underline{\text{apr}}_{R_1,\mathcal{T},\gamma}(X)$. The inclusion $\overline{\text{apr}}_{R_1,\mathcal{S},\gamma}(X) \subseteq \overline{\text{apr}}_{R_2,\mathcal{S},\gamma}(X)$ follows by duality. We conclude that (RM) is satisfied. □

The other properties are not satisfied, as illustrated in the next example.

Example 9.1.24. Let $U = \{x_1, x_2, \ldots, x_{100}\}$ and let R be the fuzzy similarity relation with $R(x_i, x_j) = 0.7$ for $i \neq j$. Consider the Łukasiewicz t-norm and t-conorm, then $(\underline{\text{apr}}_{\text{R},\mathcal{T},\gamma}(\emptyset))(x) = 0.3$ for all $x \in U$, hence, the properties (INC), (CS) and

(UE) do not hold. Since $(\overline{\text{apr}}_{R,\mathscr{S},\gamma}(\underline{\text{apr}}_{R,\mathscr{T},\gamma}(\emptyset)))(x) = 1$ for all $x \in U$, the property (LU) is not satisfied. The property (A) is not satisfied either, since $\emptyset \subseteq \underline{\text{apr}}_{R,\mathscr{T},\gamma}(\emptyset)$, but $\overline{\text{apr}}_{R,\mathscr{S},\gamma}(\emptyset) = U \nsubseteq \emptyset$. Property (ID) is not satisfied, since $(\overline{\text{apr}}_{R,\mathscr{S},\gamma}(U))(x) = 0.3$ and $(\overline{\text{apr}}_{R,\mathscr{S},\gamma}(\overline{\text{apr}}_{R,\mathscr{S},\gamma}(U)))(x) = 1$.

On the other hand, let $U = \{x, y, z\}$, R a fuzzy similarity relation such that $R(x, y) = R(x, z) = 0.4$ and $R(y, z) = 0.5$ and $\gamma = 0.6$. Consider the Łukasiewicz t-norm and t-conorm. Let $X = 0.6/x + 0.6/y + 0.2/z$ and $Y = 0.6/x + 0/y + 0.7/z$, then $(\underline{\text{apr}}_{R,\mathscr{T},\gamma}(X))(z) = 1$, $(\underline{\text{apr}}_{R,\mathscr{T},\gamma}(Y))(z) = 0.7$ and

$$(\underline{\text{apr}}_{R,\mathscr{T},\gamma}(\underline{\text{apr}}_{R,\mathscr{T},\gamma}(X \cap Y)))(z) = 0.5.$$

Hence, the property (IU) is not satisfied.

Next, we discuss the fuzzy variable precision rough set model proposed by Zhao et al., where a level of uncertainty is introduced in the approximated set.

9.2 Fuzzy variable precision rough set model

Another model designed to make approximation operators more robust, is the Fuzzy Variable Precision Rough Set (FVPRS) model, proposed by Zhao et al. [196] in 2009. It introduces a level α of uncertainty into the IC model. Below, we recall the definition of the approximation operators in this model.

Definition 9.2.1. [196] Let (U, R) be a fuzzy relation approximation space, $\mathscr{I}$ an implicator, $\mathscr{C}$ a conjunctor, $\mathscr{N}$ an involutive negator, $\mathscr{D}$ a disjunctor and $\alpha \in [0, 1)$. The *α-variable precision fuzzy rough approximation operators* $(\underline{\text{apr}}_{R,\mathscr{I},\alpha}, \overline{\text{apr}}_{R,\mathscr{C},\alpha})$ are defined by, for $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}
(\underline{\text{apr}}_{R,\mathscr{I},\alpha}(X))(x) &= \inf_{y \in U} \mathscr{I}(R(y, x), \mathscr{D}(\alpha, X(y))), \\
(\overline{\text{apr}}_{R,\mathscr{C},\alpha}(X))(x) &= \sup_{y \in U} \mathscr{C}(R(y, x), \mathscr{C}_{\mathscr{D},\mathscr{N}}(\mathscr{N}(\alpha), X(y))).
\end{aligned}$$

In [196], the following fuzzy logical connectives were considered: let $\mathscr{T}$ be a lower semi-continuous t-norm, $\mathscr{N}$ an involutive negator and $\mathscr{D}$ the maximum operator, then the implicator $\mathscr{I}$ and the conjunctor $\mathscr{C}$ are either the R-implicator of $\mathscr{T}$ and the induced conjunctor of $\mathscr{I}$ and $\mathscr{N}$ or the S-implicator based on the

$\mathcal{N}$-dual of $\mathcal{T}$ and $\mathcal{N}$ and the t-norm $\mathcal{T}$.

In the model of Zhao, elements with very small membership degrees to X are smoothed with an uncertainty level α to limit their impact on the lower approximation, while the opposite happens for the upper approximation. For this reason, it is clear that the pair $(\underline{\text{apr}}_{\text{R},\mathcal{I}}, \overline{\text{apr}}_{\text{R},\mathcal{T}}) \leq (\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$. In general, α will be chosen close to 0. Note that if α is equal to 0 and $\mathcal{D}$ is a border disjunctor, we derive the IC model as a special case of the FVPRS model.

On the other hand, we can also interpret the FVPRS model as a specific instance of the IC model. Indeed, let $X \in \mathcal{F}(U)$, then we define the fuzzy sets Y_1 and Y_2 in U by, for $x \in U$,

$$
\begin{aligned}
Y_1(x) &= \mathcal{D}(\alpha, X(x)),\\
Y_2(x) &= \mathcal{C}_{\mathcal{D},\mathcal{N}}(\mathcal{N}(\alpha), X(x)),
\end{aligned}
$$

then it is clear that $\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}(Y_1) = \underline{\text{apr}}_{\text{R},\mathcal{I}}(X)$ and $\overline{\text{apr}}_{\text{R},\mathcal{C},\alpha}(Y_2) = \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha}(X)$. Based on this relationship, it is now easy to see that (SM), (RM) and (IU) always hold for the FVPRS model, and that (D), (ID) and (LU) hold under the same conditions as for the IC model. It also satisfies (A), but under stricter conditions than compared to the ones used in the IC model.

Proposition 9.2.2. Let (U,R) be a fuzzy relation approximation space, $\mathcal{I}$ an implicator, $\mathcal{C}$ a conjunctor, $\mathcal{N}$ an involutive negator, $\mathcal{D}$ a disjunctor and $\alpha \in [0,1)$.

- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$ satisfies (D) with respect to the involutive negator $\mathcal{N}$ if $\mathcal{C}$ is the induced conjunctor of $\mathcal{I}$ and $\mathcal{N}$.
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$ satisfies (SM).
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$ satisfies (IU) if $\mathcal{D}$ is the maximum operator.
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$ satisfies (ID) if $\mathcal{C}$ is a left-continuous t-norm $\mathcal{T}$, $\mathcal{I}$ is its R-implicator and R is $\mathcal{T}$-transitive.
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\alpha}, \overline{\text{apr}}_{\text{R},\mathcal{C},\alpha})$ satisfies (ID) if $\mathcal{C}$ is a left-continuous t-norm $\mathcal{T}$, $\mathcal{I}$ is the S-implicator based on the $\mathcal{N}$-dual of $\mathcal{T}$ and $\mathcal{N}$ and R is $\mathcal{T}$-transitive.

- The pair $(\underline{\text{apr}}_{R,\mathscr{I},\alpha}, \overline{\text{apr}}_{R,\mathscr{C},\alpha})$ satisfies (LU) if $\mathscr{C}$ is a left-continuous t-norm $\mathscr{T}$, $\mathscr{I}$ is its R-implicator and R is symmetric and $\mathscr{T}$-transitive.

- The pair $(\underline{\text{apr}}_{R,\mathscr{I},\alpha}, \overline{\text{apr}}_{R,\mathscr{C},\alpha})$ satisfies (A) if $\mathscr{C}$ is an IMTL-t-norm $\mathscr{T}$, $\mathscr{I}$ is its R-implicator, $\mathscr{N}$ is the induced negator of $\mathscr{I}$, $\mathscr{D}$ is the dual of $\mathscr{T}$ with respect to the induced negator of $\mathscr{I}$ and R is symmetric.

- The pair $(\underline{\text{apr}}_{R,\mathscr{I},\alpha}, \overline{\text{apr}}_{R,\mathscr{C},\alpha})$ satisfies (RM).

Proof.

- The proof of (D) is similar to the proof of Proposition 7.3.1.

- The property (SM) holds, since every implicator, conjunctor and disjunctor is increasing in the second parameter.

- For (IU), let $X_1, X_2 \in \mathscr{F}(U)$ and define Y_1 and Y_2 by, for $i = 1, 2$,

$$\forall x \in U \colon Y_i(x) = \max(\alpha, X_i(x)),$$

then

$$\begin{aligned}
& (\underline{\text{apr}}_{R,\mathscr{I},\alpha}(X_1 \cap X_2))(x) \\
= \ & \inf_{y \in U} \mathscr{I}(R(y,x), \max(\alpha, \min(X_1(y), X_2(y)))) \\
= \ & \inf_{y \in U} \mathscr{I}(R(y,x), \min(\max(\alpha, X_1(y)), \max(\alpha, X_2(y)))) \\
= \ & \inf_{y \in U} \mathscr{I}(R(y,x), \min(Y_1(y), Y_2(y))) \\
= \ & (\underline{\text{apr}}_{R,\mathscr{I}}(Y_1 \cap Y_2))(x) \\
= \ & (\underline{\text{apr}}_{R,\mathscr{I}}(Y_1) \cap \underline{\text{apr}}_{R,\mathscr{I}}(Y_2))(x) \\
= \ & (\underline{\text{apr}}_{R,\mathscr{I},\alpha}(X_1) \cap \underline{\text{apr}}_{R,\mathscr{I},\alpha}(X_2))(x).
\end{aligned}$$

The proof for the upper approximation is similar, since $\mathscr{C}_{\mathscr{S}_M,\mathscr{N}} = \mathscr{T}_M$ holds for every involutive negator $\mathscr{N}$.

- The proof of (ID) is similar to the proof of Proposition 7.3.5.

- The proof of (LU) is similar to the proof of Proposition 7.3.6.

- Note that for $a, b \in [0,1]$: $\mathscr{C}_{\mathscr{D},\mathscr{N}}(a,b) = \mathscr{T}(a,b)$ and $\mathscr{I}(\mathscr{N}(a), b) = \mathscr{D}(a,b)$ (see [138]). Let $X, Y \in \mathscr{F}(U)$, then

$$
\begin{aligned}
& \overline{\text{apr}}_{\text{R},\mathscr{T},\alpha}(X) \subseteq Y \\
\Leftrightarrow\ & \forall x, y \in U\colon \mathscr{T}(R(y,x), \mathscr{T}(\mathscr{N}(\alpha), X(y))) \leq Y(x) \\
\Leftrightarrow\ & \forall x, y \in U\colon \mathscr{T}(\mathscr{T}(R(y,x), \mathscr{N}(\alpha)), X(y)) \leq Y(x) \\
\Leftrightarrow\ & \forall x, y \in U\colon X(y) \leq \mathscr{I}(\mathscr{T}(R(y,x), \mathscr{N}(\alpha)), Y(x)) \\
\Leftrightarrow\ & \forall x, y \in U\colon X(y) \leq \mathscr{I}(R(y,x), \mathscr{I}(\mathscr{N}(\alpha), Y(x))) \\
\Leftrightarrow\ & \forall x, y \in U\colon X(y) \leq \mathscr{I}(R(x,y), \mathscr{D}(\alpha, Y(x))) \\
\Leftrightarrow\ & X \subseteq \underline{\text{apr}}_{\text{R},\mathscr{I},\alpha}(Y).
\end{aligned}
$$

- The property (RM) holds, since every implicator is decreasing in the first parameter and every conjunctor is increasing in the first parameter.

□

Other properties do not hold for the FVPRS model, as illustrated in the next example.

Example 9.2.3. Let $U = \{x, y, z\}$ and $R = U \times U$. Consider the Łukasiewicz implicator and conjunctor, the standard negator and the maximum operator, then $(\underline{\text{apr}}_{\text{R},\mathscr{I},\alpha}(\emptyset))(x) = 0.1$ for all $x \in U$. Hence, the properties (INC), (CS) and (UE) are not satisfied.

In the following section, we discuss robust fuzzy rough set models which make use of other aggregation operators then the infimum and supremum operators.

9.3 Noise-tolerant models based on aggregation operators

A drawback of the IC model is its use of the infimum and supremum operators. The lower, respectively upper, approximation is fully determined by the worst, respectively best, value. To overcome this problem, fuzzy rough set models based on other aggregation operators have been defined. Models of this type are the

β-precision fuzzy rough set model of Fernández-Salido and Murakami [47] and the ordered weighted average based fuzzy rough set model proposed by Cornelis et al. [19]. We start with the former one.

9.3.1 β-precision fuzzy rough set model

The oldest noise-tolerant fuzzy rough set model is due to Fernández-Salido and Murakami [47], who proposed the β-Precision Fuzzy Rough Set (β-PREC) model in 2003. They tackle the noise problem by replacing the infimum and supremum operators by a β-precision quasi t-norm and t-conorm:

Definition 9.3.1. [47] Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{T}$ a t-norm, $\mathscr{S}$ a t-conorm and $\beta \in [0,1]$. Given an implicator $\mathscr{I}$ and a conjunctor $\mathscr{C}$, the *β-precision fuzzy rough approximation operators* $(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}, \overline{\text{apr}}_{\text{R},\mathscr{C},\mathscr{S}_\beta})$ are defined by, for $X \in \mathscr{F}(U)$ and $x \in U$,

$$
\begin{aligned}
(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(X))(x) &= \mathscr{T}_{\beta \atop y\in U} \left(\mathscr{I}(R(y,x),X(y))\right), \\
(\overline{\text{apr}}_{\text{R},\mathscr{C},\mathscr{S}_\beta}(X))(x) &= \mathscr{S}_{\beta \atop y\in U} \left(\mathscr{C}(R(y,x),X(y))\right).
\end{aligned}
$$

Controlled by β, the smallest elements are omitted in the calculation of the lower approximation. Analogously, the largest elements will not influence the upper approximation. Therefore, when we consider the minimum and maximum operator for $\mathscr{T}$ and $\mathscr{S}$, the approximation operators of the β-PREC model satisfy

$$(\underline{\text{apr}}_{\text{R},\mathscr{I}}, \overline{\text{apr}}_{\text{R},\mathscr{C}}) \leq (\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}, \overline{\text{apr}}_{\text{R},\mathscr{C},\mathscr{S}_\beta}).$$

On the other hand, the use of t-norms and t-conorms other than the minimum and maximum operator allows for more interaction among the arguments to be aggregated. When $\beta = 0$, all elements smaller than the average $\mathscr{I}(R(y,x),X(y))$ value (respectively, higher than the average $\mathscr{C}(R(y,x),X(y))$ value) are ignored. If $\beta = 1$, $\mathscr{T} = \mathscr{T}_M$ and $\mathscr{S} = \mathscr{S}_M$, we derive the IC model. Fernández-Salido and Murakami recommended to choose β very close to 1.

Next, we study the properties of the β-PREC model.

Proposition 9.3.2. Let (U,R) be a fuzzy relation approximation space with U finite, $\mathcal{T}$ a t-norm, $\mathcal{S}$ a t-conorm , $\mathcal{I}$ an implicator and $\mathcal{C}$ a conjunctor.

- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\mathcal{T}_\beta}, \overline{\text{apr}}_{\text{R},\mathcal{C},\mathcal{S}_\beta})$ satisfies (D) with respect to the standard negator, if $\mathcal{S}$ is the $\mathcal{N}_S$-dual of $\mathcal{T}$ and if $\mathcal{C}$ is the induced conjunctor of $\mathcal{I}$ and $\mathcal{N}_S$.
- The pair $(\underline{\text{apr}}_{\text{R},\mathcal{I},\mathcal{T}_\beta}, \overline{\text{apr}}_{\text{R},\mathcal{C},\mathcal{S}_\beta})$ satisfies (SM) and (RM).

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ are dual with respect to $\mathcal{N}_S$, it holds for all $k \in \mathbb{N} \setminus \{0,1\}$ that $\mathcal{T}^k$ and $\mathcal{S}^k$ are dual with respect to $\mathcal{N}_S$. We will prove that $\mathcal{T}_\beta$ and $\mathcal{S}_\beta$ are dual with respect to $\mathcal{N}_S$. Let $|U| = n$, $(a_1, a_2, \ldots, a_n)$ a n-tuple in $[0,1]$ and let σ be a permutation on $\{1,2,\ldots,n\}$ such that $a_{\sigma(i)}$ is the ith largest element of the tuple. Define m such that:

$$m = \max\left\{ i \in \{0,\ldots,n\} \mid i \leq (1-\beta)\cdot \sum_{j=1}^{n} a_j \right\}.$$

Now, since

$$m \leq (1-\beta)\cdot \sum_{j=1}^{n} a_j \Leftrightarrow m \leq (1-\beta)\cdot \sum_{j=1}^{n} 1-(1-a_j),$$

we omit m values to calculate $\mathcal{T}_\beta$ and we omit m values to calculate $\mathcal{S}_\beta$. Hence,

$$\begin{aligned}
\mathcal{N}_S(\mathcal{T}_\beta(a_1,a_2,\ldots,a_n)) &= 1-\mathcal{T}^{n-m}(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n-m)}) \\
&= \mathcal{S}^{n-m}(1-a_{\sigma(1)}, 1-a_{\sigma(2)}, \ldots, 1-a_{\sigma(n-m)}) \\
&= \mathcal{S}_\beta(1-a_1, 1-a_2, \ldots, 1-a_n) \\
&= \mathcal{S}_\beta(\mathcal{N}_S(a_1), \mathcal{N}_S(a_2), \ldots, \mathcal{N}_S(a_n)).
\end{aligned}$$

We obtain for $X \in \mathcal{F}(U)$ and $x \in U$

$$\begin{aligned}
(\underline{\text{apr}}_{\text{R},\mathcal{I},\mathcal{T}_\beta}(X^{\mathcal{N}_S}))^{\mathcal{N}_S}(x) &= \mathcal{N}_S\left(\mathop{\mathcal{T}_\beta}_{y\in U} (\mathcal{I}(R(y,x), \mathcal{N}_S(X(y)))) \right) \\
&= \mathop{\mathcal{S}_\beta}_{y\in U} \mathcal{N}_S(\mathcal{I}(R(y,x), \mathcal{N}_S(X(y)))) \\
&= \mathop{\mathcal{S}_\beta}_{y\in U} (\mathcal{C}(R(y,x), X(y)))
\end{aligned}$$

$$= \quad (\overline{\text{apr}}_{\text{R},\mathscr{C},\mathscr{S}_\beta}(X))(x).$$

It is very easy to see that (SM) and (RM) are satisfied by the monotonicity of the fuzzy logical connectives. □

In particular, the duality property holds if the pair $(\mathscr{I},\mathscr{C})$ consists of an S-implicator based on a t-conorm $\mathscr{S}$ and $\mathscr{N}_S$ and the $\mathscr{N}_S$-dual t-norm of $\mathscr{S}$, or if it consists of an IMTL-t-norm $\mathscr{T}$ and its R-implicator, assuming that the induced negator of $\mathscr{I}$ is the standard negator. Note that duality only holds with respect to the standard negator $\mathscr{N}_S$. Indeed, if $\mathscr{S}$ is the $\mathscr{N}$-dual t-conorm of a t-norm $\mathscr{T}$ for an involutive negator $\mathscr{N} \neq \mathscr{N}_S$, then $\mathscr{N}(\mathscr{T}_\beta(a_1,a_2,\ldots,a_n))$ is not necessarily equal to $\mathscr{S}_\beta(\mathscr{N}(a_1),\mathscr{N}(a_2),\ldots,\mathscr{N}(a_n))$.

None of the other properties is satisfied, as illustrated in the next example.

Example 9.3.3. Let $U=\{x_1,x_2,\ldots,x_{100}\}$ and R a fuzzy similarity relation with $R(x_i,x_j)=0.7$ if $i\neq j$. Consider the Łukasiewicz implicator and t-norm, the minimum and maximum operator for the aggregation and $\beta=0.95$. Then

$$(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(\emptyset))(x_1)=0.3,$$

hence, the properties (INC), (CS) and (UE) do not hold.

On the other hand, let $U=\{x,x_1,x_2,\ldots,x_{10}\}$ and $R=U\times U$. Consider a border implicator and border conjunctor, the minimum and maximum operator and $\beta=0.8$. Let $X,Y\in\mathscr{F}(U)$ with

$$\begin{aligned} X &= 1/x+0.1/x_1+1/x_2+0.3/x_3+1/x_4+0.5/x_5\\ &\quad +1/x_6+0.7/x_7+1/x_8+0.9/x_9+1/x_{10},\\ Y &= 0/x+1/x_1+0.2/x_2+1/x_3+0.4/x_4+1/x_5\\ &\quad +0.6/x_6+1/x_7+0.8/x_8+1/x_9+1/x_{10}, \end{aligned}$$

then $(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(X))(x)=0.3$, $(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(Y))(x)=0.2$, but for the intersection $X\cap Y$ we have $(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(X\cap Y))(x)=0.1$. Hence, (IU) is not satisfied.

Moreover, let $U=\{x,y,z\}$ and R a fuzzy $\mathscr{T}$-similarity relation defined by $R(x,y)=0.3$, $R(x,z)=0.6$ and $R(y,z)=0.7$ with $\mathscr{T}$ the Łukasiewicz t-norm.

Consider the Łukasiewicz implicator, t-norm and t-conorm and $\beta = 0.9$. Let $X = 0.5/x + 0.8/y + 0.3/z$ and $Y = 0.5/x + 0.9/y + 0.7/z$, then

$$\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(Y) = 0.5/x + 0.9/y + 0.6/z,$$

and thus $X \subseteq \underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(Y)$. However, since $(\overline{\text{apr}}_{\text{R},\mathscr{T},\mathscr{S}_\beta}(X))(x) = 0.6$, it does not hold that $\overline{\text{apr}}_{\text{R},\mathscr{T},\mathscr{S}_\beta}(X) \subseteq Y$. Thus, the property (A) is not satisfied. Moreover, since

$$\begin{aligned}(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(Y)))(y) &= 0.8,\\ (\overline{\text{apr}}_{\text{R},\mathscr{T},\mathscr{S}_\beta}(\underline{\text{apr}}_{\text{R},\mathscr{I},\mathscr{T}_\beta}(Y)))(y) &= 1,\end{aligned}$$

the properties (ID) and (LU) are not satisfied either.

Next, we discuss Cornelis et al.'s ordered weighted average based fuzzy rough set model.

9.3.2 Ordered weighted average based fuzzy rough set model

In 2010, Cornelis et al. [19] constructed a fuzzy rough set model in which they replaced the infimum and supremum operators of the IC model by Ordered Weighted Average (OWA) aggregation operators:

Definition 9.3.4. [19] Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ an implicator, $\mathscr{C}$ a conjunctor and OWA weight vectors W_1 and W_2 of length n, with $n = |U|$ and such that andness$(W_1) > 0.5$ and orness$(W_2) > 0.5$. The (W_1, W_2)-*fuzzy rough approximation operators* $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}, \overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2})$ are defined by, for $X \in \mathscr{F}(U)$ and $x \in U$,

$$\begin{aligned}(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X))(x) &= \underset{y\in U}{\text{OWA}_{W_1}} \langle \mathscr{I}(R(y,x), X(y))\rangle,\\ (\overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2}(X))(x) &= \underset{y\in U}{\text{OWA}_{W_2}} \langle \mathscr{C}(R(y,x), X(y))\rangle.\end{aligned}$$

By varying the OWA weight vectors, different fuzzy rough set models can be maintained. Clearly, for the weight vectors $W_1 = \langle 0, \ldots, 0, 1\rangle$ and $W_2 = \langle 1, 0, \ldots, 0\rangle$, we obtain the IC model. If other OWA weight vectors are used, more weight will be given to higher, respectively lower values, so it always holds that

$$(\underline{\text{apr}}_{\text{R},\mathscr{I}}, \overline{\text{apr}}_{\text{R},\mathscr{C}}) \leq (\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}, \overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2}).$$

A special case of the OWA model is the robust nearest neighbor fuzzy rough model, proposed in 2012 by Hu et al. [72]. For instance, with the *k-trimmed minimum* operator, the authors consider a variation on the IC lower approximation, in which the k smallest implication values $\mathcal{I}(R(y,x),X(y))$ are omitted in the infimum computation. Clearly, this approach can be modeled by using the OWA weight vector in which the element on position $n-k$ is equal to 1 and the remaining values are 0. In Table 9.1, we show for each robust nearest neighbor fuzzy rough approximation proposed in [72] its corresponding OWA weight vector. Note that the first three correspond to lower approximations, and the last three to upper approximations.

Next, we consider the properties of the OWA model. Duality holds under conditions on the weight vectors W_1 and W_2 which we discuss first. Let W_1 be a weight vector such that andness$(W_1) > 0.5$, then it holds that

$$1 - \frac{n}{n-1} \cdot 1 + \frac{1}{n-1} \sum_{i=1}^{n} i \cdot (W_1)_i > 0.5$$

and thus

$$\frac{1}{n-1} \sum_{i=1}^{n} i \cdot (W_1)_i > \frac{n}{n-1} - 0.5.$$

If we define W_2 as $(W_2)_i = (W_1)_{n-i+1}$ for $i \in \{1,2,\ldots,n\}$, then it holds that

$$\begin{aligned}
\text{orness}(W_2) &= \frac{1}{n-1} \sum_{i=1}^{n} (n-i).(W_2)_i \\
&= \frac{1}{n-1} \sum_{i=1}^{n} (n-i).(W_1)_{n-i+1} \\
&= \frac{1}{n-1} \sum_{j=1}^{n} (n-(n-j+1)).(W_1)_j \\
&= \frac{1}{n-1} \sum_{j=1}^{n} (j-1).(W_1)_j \\
&= \frac{1}{n-1} \sum_{j=1}^{n} j.(W_1)_j - \frac{1}{n-1}
\end{aligned}$$

Table 9.1: Correspondence between robust nearest neighbor fuzzy rough approximation operators and OWA weight vectors

Operator	OWA weight vector
k-trimmed minimum	$w_i = \begin{cases} 1 & \text{if } i = n-k \\ 0 & \text{otherwise} \end{cases}$
k-mean minimum	$w_i = \begin{cases} \frac{1}{k} & \text{if } i > n-k \\ 0 & \text{otherwise} \end{cases}$
k-median minimum	$w_i = \begin{cases} 1 & \text{if } k \text{ odd, } i = n - \frac{k-1}{2} \\ \frac{1}{2} & \text{if } k \text{ even, } i = n - \frac{k}{2} \text{ or } i = n - \frac{k-2}{2} \\ 0 & \text{otherwise} \end{cases}$
k-trimmed maximum	$w_i = \begin{cases} 1 & \text{if } i = k+1 \\ 0 & \text{otherwise} \end{cases}$
k-mean maximum	$w_i = \begin{cases} \frac{1}{k} & \text{if } i < k+1 \\ 0 & \text{otherwise} \end{cases}$
k-median maximum	$w_i = \begin{cases} 1 & \text{if } k \text{ odd, } i = \frac{k+1}{2} \\ \frac{1}{2} & \text{if } k \text{ even, } i = \frac{k}{2} \text{ or } i = \frac{k}{2}+1 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
&> \frac{n}{n-1} - 0.5 - \frac{1}{n-1}\\
&= 1-0.5\\
&= 0.5.
\end{aligned}$$

Hence, if andness$(W_1) > 0.5$, then orness$(W_2) > 0.5$.

Proposition 9.3.5. Let (U,R) be a fuzzy relation approximation space with U finite, $\mathscr{I}$ an implicator and $\mathscr{C}$ a conjunctor. Let W_1 be a weight vector such that andness$(W_1) > 0.5$ and W_2 be a weight vector such that orness$(W_2) > 0.5$.

- The pair $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}, \overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2})$ satisfies (D) with respect to the standard negator, if $(W_2)_i = (W_1)_{n-i+1}$ for $i \in \{1,2,\ldots,n\}$ with $|U| = n$ and $\mathscr{C}$ is the induced conjunctor of $\mathscr{I}$ and $\mathscr{N}_S$.
- The pair $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}, \overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2})$ satisfies (SM) and (RM).

Proof. Let $X \in \mathscr{F}(U)$ and $x \in U$. Since U is finite, we can rename the elements of U such that $U = \{z_1, z_2, \ldots, z_n\}$ and

$$\mathscr{C}(R(z_1,x),X(z_1)) \geq \mathscr{C}(R(z_2,x),X(z_2)) \geq \ldots \geq \mathscr{C}(R(z_n,x),X(z_n)).$$

As $\mathscr{C}$ is the induced conjunctor of $\mathscr{I}$ and $\mathscr{N}_S$, it holds for every $z_i \in U$ that

$$\mathscr{C}(R(z_i,x),X(z_i)) = 1 - \mathscr{I}(R(z_i,x), 1 - X(z_i)).$$

Hence,

$$\mathscr{I}(R(z_1,x),1-X(z_1)) \leq \mathscr{I}(R(z_2,x),1-X(z_2)) \leq \ldots \leq \mathscr{I}(R(z_n,x),1-X(z_n)).$$

Thus, for each z_i, $\mathscr{I}(R(z_i,x),1-X(z_i))$ is multiplied with weight $(W_1)_{n-i+1}$ when computing the OWA_{W_1} value. We obtain that

$$\begin{aligned}
&(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X^{\mathscr{N}_S}))^{\mathscr{N}_S}(x)\\
= \;& 1 - \left(\sum_{i=1}^{n} (W_1)_{n-i+1} \cdot \mathscr{I}(R(z_i,x), 1 - X(z_i))\right)\\
= \;& \left(\sum_{i=1}^{n} (W_1)_{n-i+1} \cdot (1 - \mathscr{I}(R(z_i,x), 1 - X(z_i)))\right)
\end{aligned}$$

$$= \left(\sum_{i=1}^{n} (W_2)_i \cdot \mathscr{C}(R(z_i, x), X(z_i))\right)$$
$$= (\overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2}(X))(x).$$

The properties (SM) and (RM) were proven in [19]. □

In particular, duality holds if the pair $(\mathscr{I}, \mathscr{C})$ consists of an S-implicator based on the t-conorm $\mathscr{S}$ and standard negator $\mathscr{N}_S$ and the $\mathscr{N}_S$-dual t-norm of $\mathscr{S}$, or if it consists of an IMTL-t-norm $\mathscr{T}$ and its R-implicator, assuming that the induced negator of the implicator equals the standard negator. None of the other properties is satisfied, as illustrated in the next example.

Example 9.3.6. Let $U = \{x, y\}$ with R a fuzzy similarity relation defined by $R(x, y) = 0.5$. Consider the Łukasiewicz implicator and t-norm and let $W_1 = \langle \frac{1}{3}, \frac{2}{3} \rangle$ and $W_2 = \langle \frac{2}{3}, \frac{1}{3} \rangle$, then $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(\emptyset))(x) = \frac{1}{6}$. The properties (INC), (CS) and (UE) are not satisfied.

On the other hand, let $U = \{x, x_1, x_2, \ldots, x_{10}\}$ and $R = U \times U$. Consider a border implicator and border conjunctor and

$$W_1 = \langle 0.10, 0.09, 0.08, 0.07, 0.06, 0.05, 0.04, 0.03, 0.02, 0.01, 0.45 \rangle,$$
$$W_2 = \langle 0.45, 0.10, 0.09, 0.08, 0.07, 0.06, 0.05, 0.04, 0.03, 0.02, 0.01 \rangle.$$

Let $X, Y \in \mathscr{F}(U)$ with

$$X = 1/x + 0.1/x_1 + 1/x_2 + 0.3/x_3 + 1/x_4 + 0.5/x_5 + 1/x_6 + 0.7/x_7 + 1/x_8 + 0.9/x_9 + 1/x_{10},$$
$$Y = 0/x + 1/x_1 + 0.2/x_2 + 1/x_3 + 0.4/x_4 + 1/x_5 + 0.6/x_6 + 1/x_7 + 0.8/x_8 + 1/x_9 + 1/x_{10},$$

then $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X))(x) = 0.565$, $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(Y))(x) = 0.51$, but for the intersection $X \cap Y$ we have $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X \cap Y))(x) = 0.385$. Hence, (IU) is not satisfied.

Next, let $U = \{x, y, z\}$ and R a fuzzy similarity relation with $R(x, y) = 0.8$, $R(x, z) = R(y, z) = 0.2$. Let $\mathscr{I}$ be the Gödel implicator and $\mathscr{C}$ the minimum operator and $W_1 = \langle 0.05, 0.05, 0.9 \rangle$ and $W_2 = \langle 0.9, 0.05, 0.05 \rangle$. Let $X = 0.7/x + 0.2/y + 0.1/z$, then $\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X) = 0.135/x + 0.135/y + 0.19/z$. Since

$$(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X)))(z) = 0.13775,$$

(ID) does not hold for a left-continuous t-norm and its implicator. Since

$$(\overline{\text{apr}}_{\text{R},\mathscr{T},\text{W}_2}(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X)))(x) = 0.1845,$$

(LU) does not hold. Let R be the fuzzy similarity relation with $R(x,y) = 1$ and $R(x,z) = R(y,z) = 0.8$ and consider the Kleene-Dienes implicator, then for the same fuzzy set X we obtain $\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X) = 0.255/x + 0.255/y + 0.135/z$. However, $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(X)))(x) = 0.2055$, hence, (ID) is not satisfied for an S-implicator based on a t-conorm $\mathscr{S}$ and a negator $\mathscr{N}$ and the $\mathscr{N}$-dual of $\mathscr{S}$.

Finally, let $U = \{x, y, z\}$ and R a fuzzy $\mathscr{T}$-similarity relation with $R(x,y) = 0.8$, $R(x,z) = 0.3$ and $R(y,z) = 0.2$, with $\mathscr{T}$ the Łukasiewicz t-norm. Let $\mathscr{I}$ be the Łukasiewicz implicator and $W_1 = \langle \frac{1}{6}, \frac{2}{6}, \frac{3}{6} \rangle$ and $W_2 = \langle \frac{3}{6}, \frac{2}{6}, \frac{1}{6} \rangle$. Consider the fuzzy sets $X = 0.7/x + 0/y + 0.1/z$ and $Y = 0.8/x + 0.3/y + 0.7/z$, then

$$\overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2}(X) = 0.35/x + 0.25/y + 0.05/z,$$

hence, $\overline{\text{apr}}_{\text{R},\mathscr{C},\text{W}_2}(X) \subseteq Y$. Since $(\underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(Y))(x) = \frac{41}{60}$, it does not hold that $X \subseteq \underline{\text{apr}}_{\text{R},\mathscr{I},\text{W}_1}(Y)$, hence, the property (A) is not satisfied.

In the following section, we empirically analyze the robustness of the different models considered in this chapter.

9.4 Experimental evaluation on robustness

To evaluate the robustness of the previously discussed fuzzy rough set models and the IC model, we conduct an experiment involving four real-world data sets. Each data set can be considered as a complete decision table

$$T = (U, At = C \cup \{d\}, \{V_a \mid a \in At\}, \{I_a \mid a \in At\})$$

where U is the finite set of objects which are described by the conditional attributes in C and one decision attribute $d \notin C$.

A common task in machine learning is to predict the value of the decision attribute of an object, given the other attributes of that object and previously labeled training data. To this end, many fuzzy rough set applications of machine learning (see e.g. [18, 159]) use the so-called positive region, a fuzzy set in U defined as follows:

$$\forall x \in U\colon\ \mathrm{POS}_A(x) = \sup_{y\in U} (\underline{\mathrm{apr}}_A(R_d^p(y)))(x),$$

with $\underline{\mathrm{apr}}_A$ a fuzzy lower approximation operator in the fuzzy relation approximation space (U, R_A), with R_A a fuzzy relation based on the set of conditional attributes $A \subseteq C$ and R_d a fuzzy relation based on the decision attribute d. The fuzzy relations R_A and R_d represent approximate equality between objects based on the conditional attributes, and on the decision attribute, respectively. They are assumed to be (at least) reflexive and symmetric. Note that for a crisp equivalence relation E_d and a crisp equivalence relation E_A, we obtain the positive region defined in Eq. (2.3): since E_d and E_A are crisp, $\mathrm{POS}_A(x)$ is either 0 or 1 and thus,

$$\forall x \in U\colon\ \mathrm{POS}_A(x) = 1 \quad \Leftrightarrow \quad x \in \mathrm{POS}_{E_A}([x]_{E_d}).$$

In this work, we use the following definition for R_A, $A \subseteq C$:

$$\forall x, y \in U\colon R_A(x,y) = \frac{\sum\limits_{a\in A} 1 - \frac{|I_a(x)-I_a(y)|}{\mathrm{range}(a)}}{|A|}$$

where range(a) equals the difference between the maximum value of the attribute a and its minimum value. The fuzzy relation R_d is defined as follows when d is a categorical (discrete) decision attribute:

$$\forall x, y \in U\colon R_d(x,y) = \begin{cases} 1 & I_d(x) = I_d(y) \\ 0 & \text{otherwise} \end{cases}$$

and as follows for a continuous decision attribute:

$$\forall x, y \in U\colon R_d(x,y) = 1 - \frac{|I_d(x) - I_d(y)|}{\mathrm{range}(d)}.$$

For a robust fuzzy rough set model, we would like the positive region not to change drastically when small changes in the data occur. This should hold both when the conditional attributes are affected by noise (*attribute noise*), as well as when the decision attribute contains errors (*class noise*). Given a certain noise level $n \in [0, 100]$, we define an altered decision system with $At^n = C^n \cup \{d\}$: each attribute a has a n% chance of having their values altered to other attribute values in the range of a. To that end, the attribute values of all objects in the decision system are considered separately. For each attribute and each object, a random number $r \in [0, 1]$ is generated. If this number r is lower than n%, the attribute value of the object is changed to a random value in the range of a. This means that in the asymptotic case of an infinite amount of data, n% of the attribute values is altered. Analogously, we can define an altered decision system with $At_n = C \cup \{d^n\}$: each decision value has an n% chance of being altered to a value in the range of d. That is, for each $x \in U$, we generate a random number $r \in [0, 1]$ and if this value is lower than n%, the value $I_d(x)$ is altered to a random value in the range of d.

In order to evaluate the robustness of the fuzzy rough models, we therefore carry out the following procedure for each data set:

1. Calculate the positive regions $\mathrm{POS}_C(x)$ of all objects $x \in U$ using the specified fuzzy rough set model.

2. For the noise levels $n = 1, 2, \ldots, 30$, calculate for each $x \in U$ the positive region $\mathrm{POS}_C^{a,n}(x)$, now on the altered decision system with At^n where each attribute value $I_a(x)$ has an n% chance of being altered to an attribute value in the range of a.

3. For the noise levels $n = 1, 2, \ldots, 30$, calculate for each $x \in U$ the positive region $\mathrm{POS}_C^{d,n}(x)$, now on the altered decision system with At_n where each decision value $I_d(x)$ has an n% chance of being altered to an attribute value in the range of d.

4. For each noise level, calculate the average distances between the original positive regions and the altered positive regions:

$$\mathrm{error}_{a,n} = \frac{\sum\limits_{x \in U} |\mathrm{POS}_C(x) - \mathrm{POS}_C^{a,n}(x)|}{|U|},$$

$$\mathrm{error}_{d,n} = \frac{\sum\limits_{x \in U} |\mathrm{POS}_C(x) - \mathrm{POS}_C^{d,n}(x)|}{|U|}.$$

5. Repeat steps 1 through 4 ten times and report the average errors over ten runs.

These errors express to what extent the fuzzy rough set model changes when a certain level of attribute or class noise is imposed on the data.

We carry out this procedure for four data sets from the KEEL[8] data set repository. Their properties are listed in Table 9.2. Note that the first two data sets, 'Appendicitis' and 'Iris'[9], have a categorical decision attribute (classification problems), while the last two, 'Diabetes' and 'machineCPU', have a continuous decision attribute (regression problems).

Table 9.2: Data sets used in the experimental evaluation

	$\lvert U\rvert$	$\lvert C\rvert$
Appendicitis	106	7
Iris	150	4
Diabetes	43	2
machineCPU	209	6

The specific parameters which we used for the different fuzzy rough set models are described in Table 9.3. We followed the parameter suggestions of the authors for their models in the corresponding papers. In some cases, multiple parameter settings were suggested, in which case we tested all of them and report the results of the models with the most stable parameter setting, highlighted in bold in Table 9.3. For the IC model, we use the Łukasiewicz implicator to calculate the lower approximation, and we maintain this choice for the other models to make the evaluation implicator independent.

In Figures 9.1 and 9.2, we show the results of the experiment. In each case, the X-axis shows the noise level, while the corresponding error can be seen on the

[8] `www.keel.es`

[9] Also available at the UCI Machine Learning Repository [3], `http://archive.ics.uci.edu/ml`

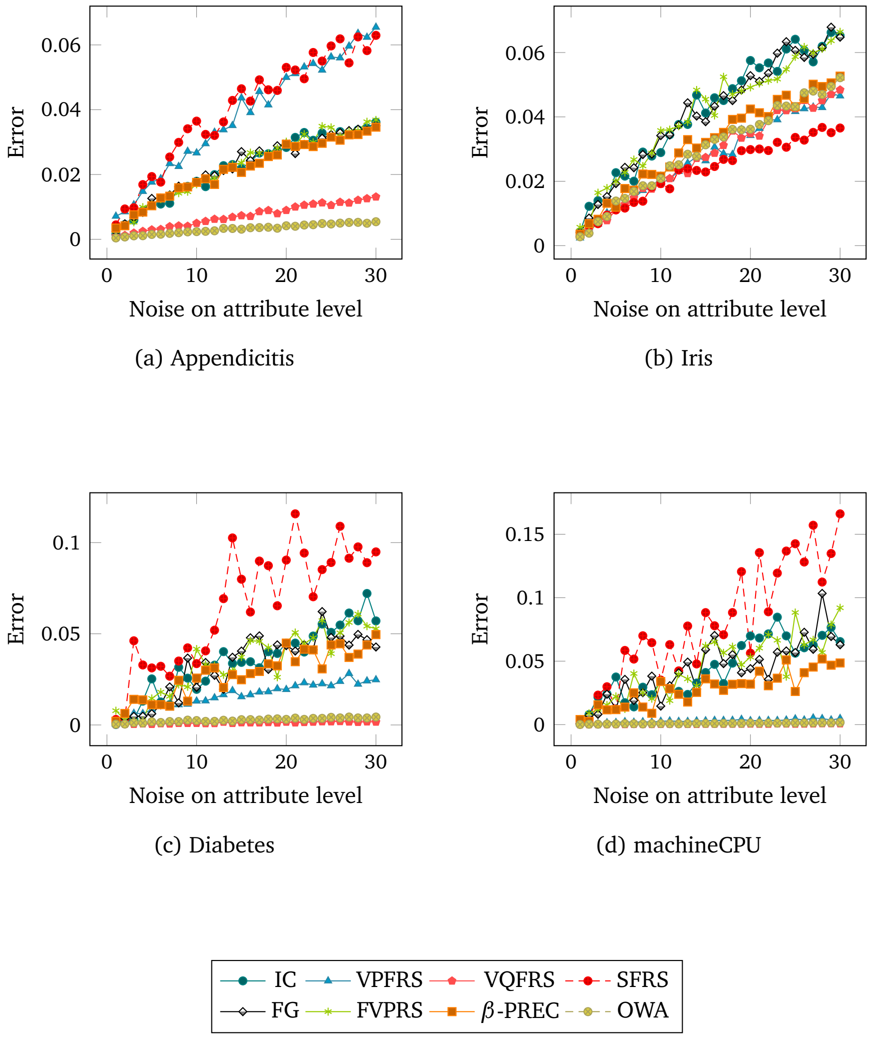


Figure 9.1: Robustness of the fuzzy rough set models to attribute noise

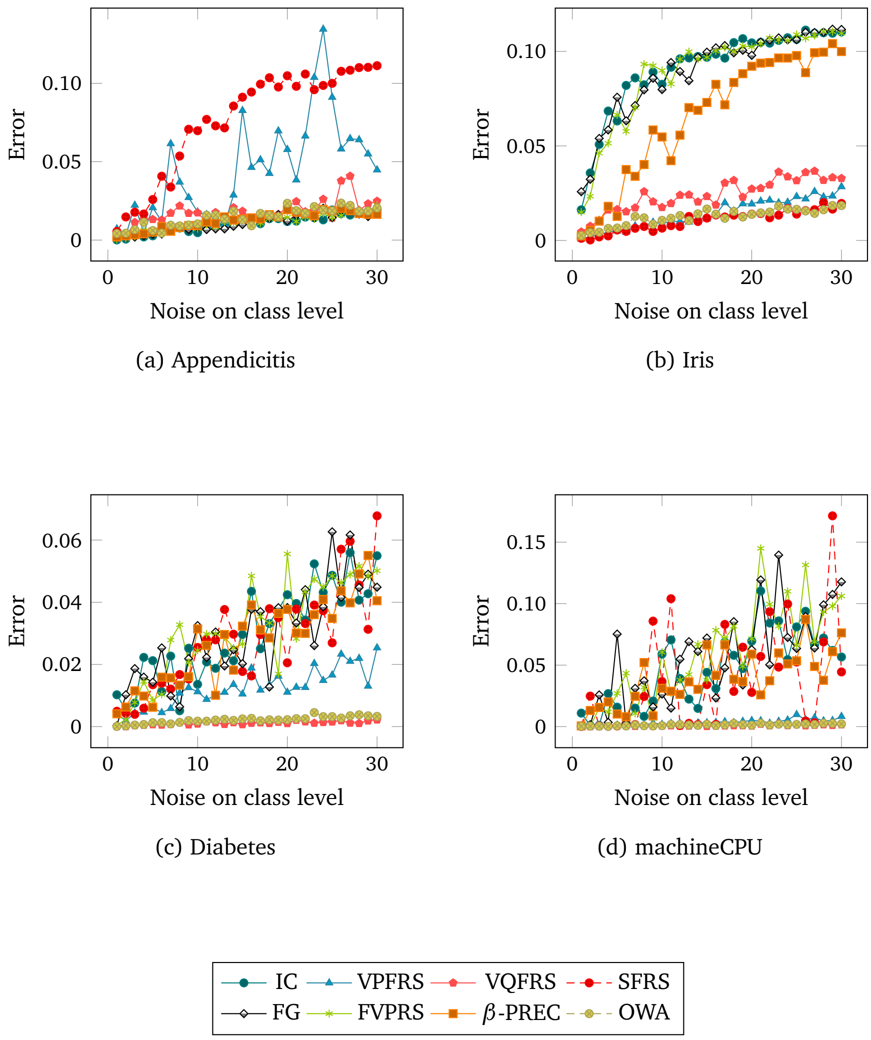


Figure 9.2: Robustness of the fuzzy rough set models to class noise

Table 9.3: Parameter settings for the fuzzy rough set models

Model	Parameters
IC	-
VPFRS	$u: 0.80, \mathbf{0.85}, 0.90$
VQFRS	$Q_{(0.2,1)}$
SFRS	$\delta: 0.10, \mathbf{0.15}, 0.20$
FG	$\gamma: \mathbf{0.80}, 0.85, 0.90$
FVPRS	$\alpha: 0.10, \mathbf{0.15}, 0.20$
β-PREC	$\beta: \mathbf{0.97}, 0.98, 0.99$
OWA	Aggregation: minimum operator Exponential weights, **Yager weights** [176] based on $Q_{(0.2,1)}$

Y-axis. Therefore, the less the increase of a curve, the better the corresponding model performs. It can be seen that the OWA model always outperforms the IC model and is in most cases the most robust model, for both attribute and class noise. The VQFRS model also performs well for most of the data sets, but it is mostly outperformed by the OWA model. The VPFRS model performs remarkably bad for the Appendicitis data set, where it is even less robust than the IC model. On the other hand, the VPFRS model is one of the most robust models for the Iris and machineCPU data sets. For all data sets, the FVPRS model and the FG model are more or less equally robust as the IC model. The β-PREC model is more robust than the IC model but is mostly outperformed by other models such as the OWA model or VQFRS. Finally, the SFRS model performs better than the OWA model for the Iris data set, but performs badly for the other data sets. Overall, the OWA model is the most robust against attribute and class noise.

9.5 Conclusions and future work

In this chapter, we have discussed seven noise-tolerant fuzzy rough set models for a fuzzy relation approximation space (U, R) with U finite. We recalled the definitions of the approximation operators and generalized, corrected or simplified them when necessary. Four models follow the objective of Ziarko's VPRS model, and only take a portion of the fuzzy set of predecessors $R^p(x)$ into account when computing the lower and upper appoximation of a fuzzy set X in the object $x \in U$. Another model adjusts the set X which we want to approximate: when computing the lower approximation operator of X, the smallest membership degrees of X are not taken into account. In machine learning techniques such as feature selection, the approximated set X is often obtained by a decision attribute d. When using the FVPRS model, we do not take objects into account with a very small membership degree to such a decision class X. Finally, we discussed two models which use other aggregation operators than the infimum and supremum operator.

For all models, we have verified which properties of Pawlak's rough set model are maintained. Besides the properties stated in Table 7.2, we discussed the (RM) property. We list an overview of the properties in Table 9.4. For the purpose of summarizing in an efficient way, we neglect potential conditions for the properties to hold in Table 9.4, since all conditions are rather reasonable when the models are used in applications.

We immediately conclude that none of the noise-tolerant models is able to retain all the properties of Pawlak's original model and its implicator-conjunctor-based extension. The FVPRS model satisfies most theoretical properties, incidentally, this is also the model that adheres most closely to the IC model.

The VQFRS model is the only model which does not satisfy (D). However, this is not necessarily a disadvantage, since in many applications only the lower approximation operator is considered.

Note that none of the noise-tolerant models satisfies (INC), a defect they share with the crisp VPRS model. This is quite remarkable, since intuitively one would expect the lower and upper approximations to be positioned on either side of the fuzzy set they are supposed to approximate. Throughout the chapter, we noticed that most noise-tolerant models provide larger lower approximations and smaller upper approximations than the IC approximation operators. However, as they do not satisfy the (INC) property, we cannot necessarily conclude that the approximations of the noise-tolerant models are more accurate than the ones of the IC model.

Table 9.4: Overview of properties for the noise-tolerant fuzzy rough set models presented in Chapter 9

Property	Satisfied by the following models:
(D)	VPFRS, SFRS, FG, FVPRS, β-PREC, OWA
(INC)	none of the models
(SM)	VPFRS, VQFRS, FG, FVPRS, β-PREC, OWA
(IU)	FVPRS
(ID)	FVPRS
(LU)	FVPRS
(CS)	none of the models
(UE)	VQFRS
(A)	FVPRS
(RM)	FG, FVPRS, β-PREC, OWA

Furthermore, the properties (SM) and (RM) are important from the application perspective. The fact that SFRS does not satisfy (SM), not even for crisp subsets of U (see [35]), makes its practical application very problematic. For instance, in a classification problem, one would expect that when a decision class gets larger, so do its approximations. On the other hand, the violation of (RM) has its own consequences for applications, like attribute selection, which consider different levels of granulation of data: indeed, when the granulation of the data imposed by the fuzzy relations R becomes finer, we expect the lower approximation not to shrink. Some attribute selection methods even rely on (RM), e.g., the QuickReduct algorithm [18] assumes that the lower approximation of classes increases when attributes are added, since adding attributes means that the membership values of the fuzzy relation increase. In this sense, the fuzzy rough set models of FG, FVPRS, β-PREC and OWA are prefered over the other ones.

The properties (IU), (ID), (LU), (CS), (UE) and (A) are less important for practical applications, and are mainly interesting from a theoretical perspective.

Next, we analyzed the robustness of the noise-tolerant fuzzy rough set models and the IC model with respect to attribute and class noise. Of the four models FG, FVPRS, β-PREC and OWA which are interesting from a theoretical point of view, we see that the models FG and FVPRS perfom more or less equally to the IC model. Hence, the lack of properties of both models is not compensated with better performance quality. The β-PREC model is more robust than the IC model, however, it is mostly outperformed by the OWA model. We can conclude that overall, the OWA model is the most robust model against class and attribute noise.

When taking both theoretical properties and experimental analysis into consideration, we would suggest that the β-PREC and OWA model are preferable when choosing a robust fuzzy rough set model. However, they still lack the (INC) property. In [32], we introduced adaptations for the β-PREC and OWA model by taking the membership degree $X(x)$ into account when computing the lower and upper approximation of X in x. The adapted β-PREC approximation operators $(\underline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{I},\mathcal{T}_\beta}, \overline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{C},\mathcal{S}_\beta})$ and adapted OWA approximation operators $(\underline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{I},\text{W}_1}, \overline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{C},\text{W}_2})$ are defined as follows: let $X \in \mathcal{F}(U)$ and $x \in U$,

$$
\begin{aligned}
(\underline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{I},\mathcal{T}_\beta}(X))(x) &= \min\left(X(x), \mathop{\mathcal{T}_\beta}_{y\in U} (\mathcal{I}(R(y,x),X(y)))\right),\\
(\overline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{C},\mathcal{S}_\beta}(X))(x) &= \max\left(X(x), \mathop{\mathcal{S}_\beta}_{y\in U} (\mathcal{C}(R(y,x),X(y)))\right),\\
(\underline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{I},\text{W}_1}(X))(x) &= \min\left(X(x), \mathop{\text{OWA}_{W_1}}_{y\in U} \langle \mathcal{I}(R(y,x),X(y))\rangle\right),\\
(\overline{\text{apr}}^{\text{a}}_{\text{R},\mathcal{C},\text{W}_2}(X))(x) &= \max\left(X(x), \mathop{\text{OWA}_{W_2}}_{y\in U} \langle \mathcal{C}(R(y,x),X(y))\rangle\right).
\end{aligned}
$$

Both adapted models satisfy more properties than their original models:

- The adapted β-PREC model satisfies (D), (SM) and (RM) such as the original β-PREC model, but also satisfies the properties (INC) and (UE).

- The adapted OWA model satisfies (D), (SM) and (RM) such as the original OWA model, but also satisfies the properties (INC) and (UE). Moreover, it satisfies (CS) when we consider a border implicator and a border conjunctor.

Additionally, we performed a similar experimental analysis on robustness. For both attribute and class noise, the adapted models perform quite equally to the original models, without a higher complexity, and they outperform the IC model. We conclude that the benefits of the original models are not lost by the adaptations which provide better theoretical properties.

A future research objective is the fusion of the results presented in Chapters 8 and 9 to a unified framework of fuzzy rough set models. More specifically, we would like to study the following topics:

- An experimental analysis on the robustness of fuzzy covering-based rough set models presented in Chapter 8 to attribute and class noise.

- Generalizing some noise-tolerant fuzzy rough set models from theoretical perspective. For instance, we could define the FG model with a general involutive negator instead of the standard negator. Moreover, we could study the models with other fuzzy neighborhoods $\mathrm{N}(x)$ than the fuzzy sets of predecessors $R^p(x)$, for $x \in U$. For example, we could study the OWA model with the fuzzy neighborhood operator $N_1^{\mathbb{C}}$, given a fuzzy covering $\mathbb{C}$.

- The study of partial order relations between different fuzzy covering-based rough set models and noise-tolerant fuzzy rough set models. For example, the FG model is a granule-based fuzzy rough set model. Therefore, we can study this model as a fuzzy covering-based rough set model.

Additionally, an important future research direction is the application of different fuzzy rough set models in machine learning, and more specifically, in applications considering feature and instance selection.

CHAPTER 10

Conclusions and future research directions

In this work, we have provided a systematic, theoretical study of rough set theory and fuzzy rough set theory from a semantical, a constructive and a practical point of view. As the research on rough set theory was very scattered, we have constructed a general framework of (fuzzy) rough set models. Within this framework, we have presented systematic investigations, providing other researchers the ability to choose between different models based on their needs.

The semantical point of view ensures us a proper and correct use of rough set theory, and it prevents potential misuses of the theory. The constructive point of view allowed us to correct mistakes in existing models, which we often generalized. Moreover, we introduced new (fuzzy) rough set models. Since rough set theory was originally introduced in order to obtain information from data, we also compared (fuzzy) rough set models with applications in mind.

Each chapter has its own contribution and can therefore be considered independently. For each topic we have stated conclusions and, if applicable, future research directions. In this chapter, we provide an overview of the most important conclusions of our research and in addition, we state different challenges for the future.

First, in Chapter 3, we have discussed three semantically sound approaches to rough set models. We have revised a semantically sound approach to Pawlak's rough set model for complete decision tables. To this aim, we constructed a descriptive language in two parts. In addition, we have introduced a semantically sound approach to covering-based rough sets. In this approach, the set of definable sets is no longer given by a Boolean algebra over a partition, which was the case for Pawlak's rough set model, but by the union-closure over a covering which is a join-semilattice. We have studied that the lower approximation operator obtained in this semantical approach coincides with the tight covering-based lower approximation operator. However, the derived upper approximation operator does not provide a definable set, but a set of definable sets. We have illustrated how this approach may lead to certain rules given a complete decision table with added semantics. Moreover, we have applied this approach to the theory of dominance-based rough set theory in which it is shown that the conceptual lower approximation operator coincides with an element-based approximation operator for a reflexive and transitive neighborhood operator. Finally, we have introduced a semantically sound approach to decision tables with missing values considering Pawlak's rough set model. We considered three types of missing values: lost values, do-not-care values and attribute-concept values. Although only the equality relation was considered, the set of definable sets is given by the union-closure over a covering. The aim of this chapter was to derive more insight in the concepts of rough set theory.

In Chapter 4, we have constructed a unified framework of dual covering-based approximation operators. We have recalled many operators studied in literature and introduced some other pairs of dual covering-based approximation operators. We have discussed equalities and partial order relations between different approximation operators in order to obtain conclusions on the accuracy ability for each pair of dual approximation operators. From a practical perspective, it is reasonable to consider the most accurate pair of approximation operators, as the lower and upper approximations of a set will be the closest to the original set. However, accuracy is not the only property to be taken into consideration. To this aim, we studied the theoretical properties for each pair of approximation operators. For applications, it is suitable to choose approximation operators which satisfy the inclusion property and the set monotonicity property. Combining both the results on accuracy and the theoretical properties, the pairs $(\underline{\text{apr}}_{N_1^{\mathbb{C}}}, \overline{\text{apr}}_{N_1^{\mathbb{C}}})$, $(\underline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}}, \overline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}})$, (s^-, s^+) and $(\underline{\text{apr}}_{N_{P_4}^{\mathbb{C}}}, \overline{\text{apr}}_{N_{P_4}^{\mathbb{C}}})$ are preferable to use in applications.

In Chapter 6, we have introduced the definitions of a fuzzy covering, the fuzzy neighborhood system of an object, the fuzzy minimal description of an object and the fuzzy maximal description of an object of the universe. In addition, four crisp neighborhood operators and six crisp coverings studied in [189] have been extended to the fuzzy setting. For a finite fuzzy covering, four fuzzy neighborhood operators and six fuzzy coverings, one original and five derived ones, have been combined, resulting in 24 combinations of fuzzy neighborhood operators. However, we have proven that for a left-continuous t-norm and its residual implicator the obtained 24 combinations can be reduced to 16 different groups of fuzzy neighborhood operators. In this setting, we have derived the Hasse diagram of these 16 groups, which expresses which operators yield larger or smaller fuzzy neighborhoods. We have discussed the properties for each of the 16 groups of fuzzy neighborhood operators. Finally, we have discussed a family of fuzzy neighborhood operators introduced by Ma in [107]. We have shown that only for the parameter $\beta = 1$ the fuzzy neighborhood operator is reflexive. We have discussed the properties of this fuzzy neighborhood operator and studied the partial order relations with the 16 fuzzy neighborhood operators discussed above.

Besides the study of rough set models, we have studied many fuzzy rough set models in the second part of this dissertation. In Chapter 7, we have presented a historical overview of fuzzy rough set theory since the late 1980s. Moreover, we have introduced an implicator-conjunctor-based fuzzy rough set model which encapsulates many fuzzy rough set models described in literature and we have studied the properties of this model when a fuzzy neighborhood operator is considered.

In Chapter 8, we have studied fuzzy covering-based rough set models which extend the tight and loose granule-based approximation operators. We have recalled three existing models and introduced two new ones which extend the tight approximation operators. Moreover, we recalled one model and introduced one model which extend the loose approximation operators. Both models are equivalent to fuzzy neighborhood-based models. For each of the seven models, we have discussed its properties. All models maintain the properties of the tight, respectively loose, approximation operators given some conditions on the used t-norm and implicator. Only the intuitive extension of the tight approximation operators does not satisfy the (CS) property. Secondly, we have discussed partial order relations with respect to $\leq$ for a finite universe U, a finite fuzzy covering $\mathbb{C}$ on U, an IMTL-t-norm $\mathscr{T}$, its R-implicator $\mathscr{I}$ and its induced negator $\mathscr{N}$. We have studied 22 pairs of fuzzy covering-based rough set models, of which 17 pairs are based on a fuzzy neighborhood operator. From this, we can conclude that the pairs $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{T}})$ and $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ provide the most accurate approximations.

Finally, in Chapter 9, we have discussed seven noise-tolerant fuzzy rough set models for a fuzzy relation approximation space (U,R) with U finite. We recalled the definitions of the approximation operators and generalized, corrected or simplified them when necessary. Four models follow the objective of Ziarko's VPRS model, and only take a portion of the fuzzy set of predecessors $R^p(x)$ into account when computing the lower and upper appoximation of a fuzzy set X in the object $x \in U$. Another model adjusts the set X which we want to approximate: when computing the lower approximation operator of X, the smallest membership degrees of X are not taken into account. To end, we have discussed two models which use other aggregation operators than the infimum and supremum operator.

For all noise-tolerant fuzzy rough set models, we have verified which properties of Pawlak's rough set model are maintained. We immediately conclude that none of the noise-tolerant models is able to retain all the properties of Pawlak's original model and its implicator-conjunctor-based extension. The FVPRS model satisfies most theoretical properties, incidentally, this is also the model that adheres most closely to the IC model.

Next, we analyzed the robustness of the noise-tolerant fuzzy rough set models and the IC model with respect to attribute and class noise. Of the four models FG, FVPRS, β-PREC and OWA which are interesting from a theoretical point of view, we see that the models FG and FVPRS perfom more or less equally to the IC model. Hence, the lack of properties of both models is not compensated with better performance quality. The β-PREC model is more robust than the IC model, however, it is mostly outperformed by the OWA model.

When taking both theoretical properties and experimental analysis into consideration, we would suggest that the β-PREC and OWA model are preferable when choosing a robust fuzzy rough set model. However, they still lack the inclusion property. In [32], we introduced adaptations for the both models by taking the membership degree $X(x)$ into account when computing the lower and upper approximation of X in x. We have discussed that the adapted models satisfy more theoretical properties than the original models, without a higher complexity, and they outperform the IC model. Hence, the benefits of the original models are not lost by the adaptations which provide better theoretical properties.

There are still many challenges left. From a semantical point of view, the most important future research directions are the study of semantically sound approaches to covering-based rough set models for incomplete decision tables and to different fuzzy rough set models for both complete and incomplete decision tables. Moreover, we want to study rule induction based on upper approximation operators in order to obtain possible rules.

Next, as discussed in Chapter 4, there are many future research directions in the study of covering-based rough set theory from a computational point of view. However, the main challenge in the study of (covering-based) rough set models is their application in machine learning techniques such as feature and instance selection.

To this aim, we need to study reduction for covering-based rough set models. In [114], Miao et al. discussed relative reducts in consistent and inconsistent decision tables for Pawlak's rough set model. They considered three types of relative reducts: region preservation reducts, decision preservation reducts and relationship preservation reducts. All three are equivalent for consistent decision tables, but this statement is not maintained for an inconsistent decision table. Therefore, future work will include the study of these three types of relative reducts for covering-based rough set models.

Moreover, we will study bireducts for covering-based rough set models. Ślęzak and Janusz [148] introduced the notion of decision bireducts for Pawlak' s rough set model inspired on the methodology of biclustering [117]. In this approach, both a subset of conditional attributes which describes the decision classes and a subset of objects of the universe for which such a description is valid are selected.

In addition, we want to study which pairs of covering-based rough set models are robust against noise in the data, and how we can improve covering-based rough set models with this property in mind.

Future work regarding fuzzy rough set models includes the fusion of the results presented in Chapters 8 and 9 to a unified framework of fuzzy rough set models. First, we want to study the comparability between the approximation operators studied in Chapters 8 and 9. Next, we want to perform an experimental analysis on the robustness of fuzzy covering-based rough set models presented in Chapters 8 and 9 to attribute and class noise.

Other future research directions include the study of fuzzy extensions of other pairs of covering-based rough set approximation operators, e.g., the approximation operators of the framework of Yang and Li. Moreover, we want to generalize some noise-tolerant fuzzy rough set models from theoretical perspective. For instance, we could define the FG model with a general involutive negator instead of the standard negator. Moreover, we could study the models with other fuzzy neighborhoods $\mathrm{N}(x)$ than the fuzzy sets of predecessors $R^p(x)$, for $x \in U$. For example, we could study the OWA model with the fuzzy neighborhood operator $N_1^{\mathbb{C}}$, given a fuzzy covering $\mathbb{C}$.

Additionally, an important future research objective is the application of different fuzzy rough set models in machine learning, and more specifically, in applications considering feature and instance selection. Finally, a major topic in the machine learning community is the scaling of techniques to big data [204]. Recently, a first distributed approach to calculate fuzzy rough lower and upper approximations was presented in [2].

Appendices

APPENDIX A

Counterexamples for Chapter 4

Counterexample 1

Let $(U, \mathbb{C})_1$ be a covering approximation space with $U = \{1, 2, 3, 4\}$ and

$$\mathbb{C} = \{1, 12, 13, 24, 34, 123, 234\}.$$

The neighborhoods of the elements of U for the different neighborhood operators is presented in Table A.1. Note that we use the abbreviation, i.e., $N_1^{\mathbb{C}}(1) = \{1\}$ is denoted by '1' in the table.

Table A.1: Neighborhoods for $(U, \mathbb{C})_1$

N	1	2	3	4
a	1	2	3	4
b	1	2	3	4
c	1	124	134	234
d	1	2	3	4
e	123	124	134	234
f	123	23	23	234
g	1	2	3	4
h	123	124	134	234
i	123	23	23	234
j	123	1234	1234	234
k	123	1234	1234	234
l	1234	1234	1234	1234
m	1234	1234	1234	1234
a^{-1}	1	2	3	4
b^{-1}	1	2	3	4
c^{-1}	123	24	34	234
d^{-1}	1	2	3	4
e^{-1}	123	124	134	234
f^{-1}	1	1234	1234	4
i^{-1}	1	1234	1234	4
k^{-1}	123	1234	1234	234
l^{-1}	1234	1234	1234	1234

Counterexample 2

Let $(U,\mathbb{C})_2$ be a covering approximation space with $U = \{1,2,3,4\}$ and

$$\mathbb{C} = \{12, 23, 14\}.$$

The neighborhoods of the elements of U for the different neighborhood operators is presented in Table A.2. Note that we use the abbreviation, i.e., $N_1^{\mathbb{C}}(1) = \{1\}$ is denoted by '1' in the table.

Table A.2: Neighborhoods for $(U,\mathbb{C})_2$

N	1	2	3	4	N	1	2	3	4
a	1	2	23	14	a^{-1}	14	23	3	4
b	14	23	23	14	b^{-1}	14	23	23	14
c	124	123	23	14	c^{-1}	124	123	23	14
d	1	2	23	14	d^{-1}	14	23	3	4
e	124	123	23	14	e^{-1}	124	123	23	14
f	1	2	23	14	f^{-1}	14	23	3	4
g	14	23	23	14					
h	124	123	23	14					
i	1	2	23	14	i^{-1}	14	23	3	4
j	124	123	23	14					
k	1234	1234	23	14	k^{-1}	124	123	123	124
l	12	12	123	124	l^{-1}	1234	1234	3	4
m	1234	1234	123	124					

Counterexample 3

Let $(U,\mathbb{C})_3$ be a covering approximation space with $U = \{1,2,3,4\}$ and

$$\mathbb{C} = \{1,2,12,23,14\}.$$

The neighborhoods of the elements of U for the different neighborhood operators is presented in Table A.3. Note that we use the abbreviation, i.e., $N_1^{\mathbb{C}}(1) = \{1\}$ is denoted by '1' in the table.

Table A.3: Neighborhoods for $(U,\mathbb{C})_3$

N	1	2	3	4	N	1	2	3	4
a	1	2	23	14	a^{-1}	14	23	3	4
b	14	23	23	14	b^{-1}	14	23	23	14
c	1	2	23	14	c^{-1}	14	23	3	4
d	14	23	23	14	d^{-1}	14	23	23	14
e	124	123	23	14	e^{-1}	124	123	23	14
f	1	2	23	14	f^{-1}	14	23	3	4
g	14	23	23	14					
h	14	23	23	14					
i	1	2	23	14	i^{-1}	14	23	3	4
j	124	123	23	14					
k	1234	1234	23	14	k^{-1}	124	123	123	124
l	12	12	123	124	l^{-1}	1234	1234	3	4
m	1234	1234	123	124					

Counterexample 4

Let $(U,\mathbb{C})_4$ be a covering approximation space with $U = \{1,2,3\}$ and

$$\mathbb{C} = \{1,2,123\}.$$

The neighborhoods of the elements of U for the different neighborhood operators is presented in Table A.4. Note that we use the abbreviation, i.e., $N_1^{\mathbb{C}}(1) = \{1\}$ is denoted by '1' in the table.

Table A.4: Neighborhoods for $(U,\mathbb{C})_4$

N	1	2	3	N	1	2	3
a	1	2	123	a^{-1}	13	23	3
b	123	123	123	b^{-1}	123	123	123
c	1	2	123	c^{-1}	13	23	3
d	123	123	123	d^{-1}	123	123	123
e	1	2	123	e^{-1}	13	23	3
f	123	123	123	f^{-1}	123	123	123
g	123	123	123				
h	123	123	123				
i	123	123	123	i^{-1}	123	123	123
j	123	123	123				
k	123	123	123	k^{-1}	123	123	123
l	123	123	123	l^{-1}	123	123	123
m	123	123	123				

Counterexample 5

Let $(U, \mathbb{C})_5$ be a covering approximation space with $U = \{1, 2, 3, 4\}$ and

$$\mathbb{C} = \{1, 3, 13, 24, 34, 14, 234\}.$$

The upper approximations, referred to by the numbers of the pairs, of the set X with $X \in \{1, 2, 3, 4, 12, 34, 123\}$ are presented in Tables A.5 and A.6. Note that we use the abbreviation, i.e., $\overline{\text{apr}}_a(\{1\}) = \{1\}$ is denoted by '1'. Moreover, the topology induced by $\mathbb{C}$ is given by

$$\mathscr{T} = \{\emptyset, 1234, 1, 3, 13, 24, 34, 14, 234, 4, 124, 134\}.$$

Table A.5: Approximations for $(U, \mathbb{C})_5$ and different $X \subseteq U$

Pair	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{3,4\}$	$\{1,2,3\}$
1	1	2	3	24	12	234	123
2	1	24	3	4	124	34	1234
3	1	24	3	24	124	234	1234
4	1	24	3	24	124	234	1234
5	14	24	34	24	124	234	1234
6	1	24	3	1234	124	1234	1234
7	1	2	3	1234	12	1234	123
8	14	24	34	4	124	34	1234
9	134	24	134	1234	1234	1234	1234
10	134	24	134	1234	1234	1234	1234
11	1	2	23	24	12	234	123
12	1	234	3	4	1234	34	1234
13	1	24	3	24	124	234	1234
14	14	24	34	1234	124	1234	1234
15	1	2	1234	1234	12	1234	1234
16	134	234	34	34	1234	34	1234
17	134	234	1234	1234	1234	1234	1234
18	134	234	1234	1234	1234	1234	1234
19	134	234	1234	1234	1234	1234	1234
20	1234	1234	1234	1234	1234	1234	1234
21	1234	1234	1234	1234	1234	1234	1234
22	1234	1234	1234	1234	1234	1234	1234

Table A.6: Approximations for $(U, \mathbb{C})_5$ and different $X \subseteq U$

Pair	{1}	{2}	{3}	{4}	{1,2}	{3,4}	{1,2,3}
23	1	2	3	24	12	234	1234
24	1	2	3	24	12	1234	1234
25	1	2	23	24	1234	1234	1234
26	1	2	1234	1234	1234	1234	1234
27	1	24	3	4	1234	34	1234
28	1	24	3	1234	124	34	1234
29	1	234	3	1234	1234	34	1234
30	1	2	3	4	12	34	123
31	1	2	3	4	12	34	123
32	1	24	3	24	124	234	1234
33	1	24	3	24	124	234	1234
34	1	24	3	24	124	234	1234
35	1	$\emptyset$	3	24	1	234	13
36	1	2	3	1234	12	34	123

Counterexample 6

Let $(U,\mathbb{C})_6$ be a covering approximation space with $U = \{1,2,3,4,5\}$ and

$$\mathbb{C} = \{12, 234, 45\}.$$

The upper approximations, referred to by the numbers of the pairs, of the set X with $X \in \{1,2\}$ are presented in Tables A.7. Note that we use the abbreviation, i.e., $\overline{\text{apr}}_a(\{1\}) = \{1\}$ is denoted by '1' in the table. Moreover, the topology induced by $\mathbb{C}$ is given by

$$\mathscr{T} = \{\emptyset, 12345, 12, 234, 45, 2, 4, 1234, 1245, 2345, 124, 245\}.$$

Table A.7: Approximations for $(U,\mathbb{C})_6$ and different $X \subseteq U$

Pair	1	2	Pair	1	2
1	1	123	12	12	2
2	12	2	13	12	1234
3	1	123	14	12	1234
4	12	2	15	1	123
5	12	1234	16	12	2
6	12	1234	17	12	1234
7	1	123	18	12	1234
8	12	2	19	12	1234
9	12	1234	20	1	12345
10	12	1234	21	1234	234
11	1	123	22	1234	12345
32	12	123			
33	123	123			
34	12345	12345			

APPENDIX B

Properties of $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$

Let N be a neighborhood operator on the universe U. We discuss the properties of the element-based pair of approximation operators $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$.

- The pair $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$ satisfies (D).

  Proof. Let $X \subseteq U$, then

  $$\begin{aligned}(\underline{\text{apr}}_{\text{N}}(X^{\text{c}}))^{\text{c}} &= \{x \in U \mid \text{N}(x) \not\subseteq X^{\text{c}}\} \\ &= \{x \in U \mid \text{N}(x) \cap X \neq \emptyset\} \\ &= \overline{\text{apr}}_{\text{N}}(X).\end{aligned}$$

  We conclude that the pair $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$ satisfies (D). □

- The pair $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$ satisfies (INC) if and only if N is reflexive.

  Proof. Assume $(\underline{\text{apr}}_{\text{N}}, \overline{\text{apr}}_{\text{N}})$ satisfies (INC), then for $x \in U$ it holds that $\{x\} \subseteq \overline{\text{apr}}_{\text{N}}(\{x\})$, hence $\text{N}(x) \cap \{x\} \neq \emptyset$. We conclude that $x \in \text{N}(x)$ for all $x \in U$, i.e., N is reflexive.

On the other hand, assume N is reflexive and let $X \subseteq U$. For $x \in X$ it holds that $x \in \mathrm{N}(x) \cap X$, hence, $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. Thus, $X \subseteq \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. The inclusion $\underline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq X$ holds by duality. We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (INC). □

- The pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (SM).

Proof. Immediately from the definition. □

- The pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (IU).

Proof. Let $X, Y \subseteq U$, then

$$\begin{aligned} \underline{\mathrm{apr}}_{\mathrm{N}}(X \cap Y) &= \{x \in U \mid \mathrm{N}(x) \subseteq X \cap Y\} \\ &= \{x \in U \mid \mathrm{N}(x) \subseteq X \wedge \mathrm{N}(x) \subseteq Y\} \\ &= \{x \in U \mid \mathrm{N}(x) \subseteq X\} \cap \{x \in U \mid \mathrm{N}(x) \subseteq Y\} \\ &= \underline{\mathrm{apr}}_{\mathrm{N}}(X) \cap \underline{\mathrm{apr}}_{\mathrm{N}}(Y). \end{aligned}$$

The equality $\overline{\mathrm{apr}}_{\mathrm{N}}(X \cup Y) = \overline{\mathrm{apr}}_{\mathrm{N}}(X) \cup \overline{\mathrm{apr}}_{\mathrm{N}}(Y)$ can be proven similarly. We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (IU). □

- The pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (ID) if and only if N is transitive.

Proof. Assume $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies the property (ID) and let $x, y, z \in U$ such that $x \in \mathrm{N}(y)$ and $y \in \mathrm{N}(z)$. Then $y \in \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})$, and therefore $\mathrm{N}(z) \cap \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}) \neq \emptyset$. Thus, $z \in \overline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})) \subseteq \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})$. Hence, $x \in \mathrm{N}(z)$. We conclude that N is transitive.

On the other hand, assume N is transitive and $X \subseteq U$. Let $x \in U$ such that $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(X))$, then there exists $y \in \mathrm{N}(x)$ such that $y \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. As $\mathrm{N}(y) \cap X \neq \emptyset$, there exists $z \in \mathrm{N}(y)$ such that $z \in X$. As N is transitive, it holds that $z \in \mathrm{N}(x)$. Therefore, $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. The other inclusion $\underline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(\underline{\mathrm{apr}}_{\mathrm{N}}(X))$ holds by duality. We conlude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (ID). □

- If N is symmetric, then the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (LU) if and only if N is transitive.

  Proof. Assume N symmetric and the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (LU). Let $x, y, z \in U$ with $x \in \mathrm{N}(y)$ and $y \in \mathrm{N}(z)$. For $\{x\} \subseteq U$ it holds that

  $$\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}) \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})).$$

  As $\mathrm{N}(y) \cap \{x\} \neq \emptyset$, it holds that $y \in \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})$, thus, $y \in \underline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}))$. Hence, $\mathrm{N}(y) \subseteq \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})$. As N is symmetric, it holds that $z \in \mathrm{N}(y)$. Therefore, $z \in \overline{\mathrm{apr}}_{\mathrm{N}}(\{x\})$, i.e., $x \in \mathrm{N}(z)$. We conclude that N is transitive.

  On the other hand, assume N symmetric and transitive and let $X \subseteq U$. Take $x \in U$ such that $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. We need to prove that $x \in \underline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(X))$, i.e., $\mathrm{N}(x) \subseteq \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. Take $y \in \mathrm{N}(x)$. As $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$, there exists a $z \in \mathrm{N}(x)$ such that $z \in X$. As N is symmetric, $x \in \mathrm{N}(z)$ and as N is transitive, $y \in \mathrm{N}(z)$. Thus, $z \in \mathrm{N}(y)$ and $y \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. Therefore, $\overline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(\overline{\mathrm{apr}}_{\mathrm{N}}(X))$. The inclusion $\overline{\mathrm{apr}}_{\mathrm{N}}(\underline{\mathrm{apr}}_{\mathrm{N}}(X)) \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(X)$ follows by duality. We conclude that the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (LU). □

- The pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (UE).

  Proof. Immediately from the definition. □

- The pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (A) if and only if N is symmetric.

  Proof. Assume that the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (A). Let $x, y \in U$ with $x \in \mathrm{N}(y)$. For $X = \{x\}$ and $Y = \mathrm{N}(x)$, we obtain that

  $$\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}) \subseteq \mathrm{N}(x) \Leftrightarrow \{x\} \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(\mathrm{N}(x)),$$

  i.e.,

  $$\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}) \subseteq \mathrm{N}(x) \Leftrightarrow \mathrm{N}(x) \subseteq \mathrm{N}(x).$$

  As the right-hand-side of that equivalence is always true, it holds that $\overline{\mathrm{apr}}_{\mathrm{N}}(\{x\}) \subseteq \mathrm{N}(x)$, i.e., $\{z \in U \mid x \in \mathrm{N}(z)\} \subseteq \mathrm{N}(x)$. Since $x \in \mathrm{N}(y)$, it holds that $y \in \mathrm{N}(x)$. We conclude that N is symmetric.

  On the other hand, assume N symmetric and let $X, Y \subseteq U$.

- Assume $\overline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq Y$ and let $x \in X$. For $y \in \mathrm{N}(x)$ it holds that $x \in \mathrm{N}(y)$ since N is symmetric, and thus, $y \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. Hence, $y \in Y$. Therefore, $X \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(Y)$.
- Assume $X \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(Y)$ and let $x \in \overline{\mathrm{apr}}_{\mathrm{N}}(X)$. Thus, there exists $y \in X$ with $y \in \mathrm{N}(x)$. As $y \in X$, $y \in \underline{\mathrm{apr}}_{\mathrm{N}}(Y)$, i.e., $\mathrm{N}(y) \subseteq Y$. By symmetry, $x \in \mathrm{N}(y)$ and thus $x \in Y$. Therefore, $\overline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq Y$.

We conclude that

$$\overline{\mathrm{apr}}_{\mathrm{N}}(X) \subseteq Y \Leftrightarrow X \subseteq \underline{\mathrm{apr}}_{\mathrm{N}}(Y),$$

i.e., the pair $(\underline{\mathrm{apr}}_{\mathrm{N}}, \overline{\mathrm{apr}}_{\mathrm{N}})$ satisfies (A). □

Samenvatting

De manier waarmee we met data omgaan is recent veranderd. Vanwege de technologische evolutie zijn we in staat steeds meer en meer data te verwerken, en dit vaak in real time. Data is zelf een product geworden. Denk maar aan de aanbevelingen die je krijgt op streamingwebsites zoals Spotify en Netflix of op sociale media zoals Facebook en Twitter. Maar ook in de medische wereld, de banksector en door de overheid wordt data gebruikt om beslissingen te nemen.

In deze thesis presenteren we een systematische, theoretische studie rond ruwverzamelingen en vaagruwverzamelingen. We beschouwen zowel een semantisch, een computationeel, als een praktisch standpunt.

Ruwverzamelingenleer is geïntroduceerd door Pawlak [128] en heeft als doel informatie en kennis te halen uit data. Het basisidee is het benaderen van een onvolledig gekend concept door middel van een onder- en bovenbenaderingsoperator gebaseerd op een relatie die de mate waarin objecten niet te onderscheiden zijn beschrijft. De onderbenadering bevat deze objecten die zeker tot het concept behoren, terwijl de bovenbenadering deze objecten bevat die mogelijk tot het concept behoren. Vermits ruwverzamelingenleer ontworpen is om kwalitatieve of discrete data te verwerken, heeft het een beperkte toepasbaarheid voor data met reële waarden. Daarom is het zinvol om vaagruwverzamelingenleer te beschouwen. Dit is een hybride theorie van ruwverzamelingenleer en vaagverzamelingenleer. Dit laatste is geïntroduceerd door Zadeh [193] met als doel om vage concepten te beschrijven. Inleidende begrippen omtrent ruw- en vaagruwverzamelingenleer worden respectievelijk besproken in Hoofdstuk 2 en Hoofdstuk 5.

In Hoofdstuk 3 bespreken we drie benaderingen van ruwverzamelingmodellen vanuit semantisch oogpunt. We beschrijven eerst het ruwverzamelingmodel van Pawlak voor een volledige beslissingstabel. Hiervoor stellen we een beschrijvende taal op in twee delen. Daarnaast introduceren we een semantische benadering voor coveringgebaseerde ruwverzamelingen. We besluiten dat de verzameling van 'definable sets' niet langer gegeven wordt door een boolse algebra gebaseerd op een partitie, maar door de uniesluiting van een covering. We illustreren dit door deze semantische aanpak toe te passen op dominantiegebaseerde ruwverzamelingen. Ten slotte bespreken we een semantische benadering van het model van Pawlak voor een beslissingstabel met ontbrekende waarden. Ondanks het gebruik van de gelijkheidsrelatie besluiten we dat de verzameling van 'definable sets' gegeven wordt door de uniesluiting van een covering. Het doel van dit hoofdstuk is om meer inzicht te verwerven in de concepten van de ruwverzamelingenleer.

In Hoofdstuk 4 bouwen we een uniform kader op voor paren van duale coveringgebaseerde benaderingsoperatoren. We beschrijven verschillende operatoren uit de literatuur en introduceren zelf enkele operatoren. We bespreken gelijkheden en orderelaties tussen benaderingsoperatoren. Uit dit framework kunnen we besluiten trekken in verband met de accuraatheid van paren van duale benaderingsoperatoren. Vanuit praktisch oogpunt is het aanbevolen om benaderingsoperatoren te beschouwen met een hoge accuraatheid, vermits deze operatoren benaderingen opleveren die dicht bij het oorspronkelijke concept liggen. Accuraatheid is echter niet de enige karakteristiek die in acht dient genomen te worden. Daarom bestuderen we ook de theoretische eigenschappen van de benaderingsoperatoren. Zo zijn de eigenschappen van inclusie en verzamelingmonotoniciteit belangrijk voor toepassingen. Wanneer we de resultaten omtrent de accuraatheid en de theoretische eigenschappen combineren, besluiten we dat de paren $(\underline{\text{apr}}_{N_1^{\mathbb{C}}}, \overline{\text{apr}}_{N_1^{\mathbb{C}}})$, $(\underline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}}, \overline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}})$, (s^-, s^+) en $(\underline{\text{apr}}_{N_{P_4}^{\mathbb{C}}}, \overline{\text{apr}}_{N_{P_4}^{\mathbb{C}}})$ het meest geschikt zijn voor toepassingen.

In Hoofdstuk 6 introduceren we de definities van een vaagcovering, het vaagomgevingssysteem van een element, de vage minimale beschrijving van een element en de vage maximale beschrijving van een object. Daarnaast breiden we vier omgevingsoperatoren en zes coverings besproken in [189] uit naar de vaagverzamelingenleer. Gegeven een eindige vaagcovering combineren we vier vaagomgevingsoperatoren en zes vaagcoverings, de oorspronkelijke vaagcovering en vijf afgeleide vaagcoverings. De 24 vaagomgevingsoperatoren kunnen we terugbrengen tot 16 groepen van vaagomgevingsoperatoren als we een linkscontinue t-norm en de residuele implicator beschouwen. We bestuderen orderelaties tussen deze 16 groepen en bespreken de theoretische eigenschappen van vaagomgevingsoperatoren voor elke groep. Ten slotte bespreken we een familie van vaagomgevingsoperatoren die geïntroduceerd is in [107]. Echter, enkel voor $\beta = 1$ bekomen we een reflexieve vaagomgevingsoperator. We bestuderen de eigenschappen van deze vaagomgevingsoperator en onderzoeken orderelaties met de 16 groepen van vaagomgevingsoperatoren die we hierboven bespraken.

In het tweede deel van de thesis bespreken we verschillende vaagruwverzamelingmodellen. In Hoofstuk 7 geven we een historisch overzicht van vaagruwverzamelingenleer sinds de jaren '80. Verder introduceren we het implicatorconjunctorgebaseerde (IC) vaagruwverzamelingmodel. Dit model omvat verschillende modellen die beschreven zijn in de literatuur. Daarnaast bestuderen we de theoretische eigenschappen van het IC-model.

In Hoofdstuk 8 bestuderen we vaagcoveringgebaseerde ruwverzamelingmodellen die een uitbreiding zijn van de 'tight' en 'loose' coveringgebaseerde benaderingsoperatoren. We bespreken drie bestaande modellen en introduceren twee nieuwe modellen die de 'tight' benaderingsoperatoren uitbreiden. Verder bespreken we één bestaand en introduceren we één nieuw model die de 'loose' benaderingsoperatoren uitbreiden. Beide modellen zijn equivalent aan vaagruwverzamelingmodellen die gebruik maken van een vaagomgevingsoperator. We bestuderen de theoretische eigenschappen voor elk van de zeven modellen. Daarnaast bespreken we orderelaties tussen 22 verschillende vaagruwverzamelingmodellen die gebaseerd zijn op een eindige vaagcovering, een IMTL-t-norm, de residuele implicator en de geïnduceerde negator. We besluiten dat de paren van duale benaderingsoperatoren $(\underline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{I}}, \overline{\text{apr}}_{N_1^{\mathbb{C}},\mathscr{T}})$ en $(\underline{\text{apr}}'_{\mathbb{C},\text{Wu}}, \overline{\text{apr}}'_{\mathbb{C},\text{Wu}})$ de meest accurate benaderingen opleveren.

In Hoofdstuk 9 bestuderen we enkele vaagruwverzamelingmodellen die geïntroduceerd zijn in de literatuur om met ruis in data om te gaan. Voor elk model bespreken we de benaderingsoperatoren en veralgemenen, corrigeren of vereenvoudigen we het indien nodig. Vier modellen gebruiken hetzelfde idee als Ziarko gebruikte voor het VPRS-model [203] en nemen enkel een fractie van de vaagverzameling $R^p(x)$, $x \in U$, in beschouwing bij het bepalen van de onder- en bovenbenadering in x. Een ander model past de vaagverzameling aan die benaderd wordt. Verder bespreken we twee modellen die andere aggregatieoperatoren gebruiken dan het infimum en het supremum.

Voor elk model bestuderen we de theoretische eigenschappen. Geen enkel model voldoet aan alle eigenschappen die voldaan zijn door het IC-model. Het FVPRS-model dat het dichtst aanleunt tegen het IC-model voldoet aan de meeste eigenschappen. Daarnaast zijn de modellen FG, β-PREC en OWA interessant vanuit computationeel oogpunt.

Bovendien analyseren we de robuustheid van de zeven modellen en het IC-model ten opzicht van ruis in de data. De robuustheid van de modellen FG en FVPRS is vergelijkbaar met dat van het IC-model. Het gebrek aan theoretische eigenschappen wordt dus niet gecompenseerd door een betere prestatie in de experimenten. Het β-PREC-model is robuuster dan het IC-model, maar het OWA-model overtreft meestal de prestatie van het β-PREC-model en is in het algemeen het best bestand tegen ruis.

Tot slot bespreken we in Hoofdstuk 10 de belangrijkste conclusies van deze thesis en enkele toekomstige onderzoeksvragen.

List of publications

Papers in international journals listed in Science Citation Index

- L. D'eer, C. Cornelis, A comprehensive study of fuzzy covering-based rough set models: definitions, properties and interrelationships, Submitted
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- L. D'eer, C. Cornelis, Y. Y. Yao, A semantically sound approach to Pawlak rough sets and covering-based rough sets, International Journal of Approximate Reasoning 78 (2016) 62–72
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