Triangle Algebras: towards an Axiomatization of Interval-Valued Residuated Lattices

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Abstract. In this paper, we present triangle algebras: residuated lattices equipped with two modal, or approximation, operators and with a third angular point u, different from 0 (false) and 1 (true), intuitively denoting ignorance about a formula's truth value. We prove that these constructs, which bear a close relationship to several other algebraic structures including rough approximation spaces, provide an equational representation of interval-valued residuated lattices, which are triangularizations of residuated lattices; as an important case in point, we consider $\mathcal{L}^I$, the lattice of closed intervals of $[0, 1]$. As we will argue, the representation by triangle algebras serves as a crucial stepping stone to the construction of formal interval-valued fuzzy logics, and in particular to the axiomatic formalization of residuated t-norm based logics on $\mathcal{L}^I$, in a similar way as was done for formal fuzzy logics on the unit interval.

1 Introduction and Preliminaries

Formal fuzzy logics (also: fuzzy logics in the narrow sense) are generalizations of classical logic that allow us to reason gradually. Indeed, in the scope of these logics, formulas can be assigned not only 0 and 1 as truth values, but also elements of [0,1], or, more generally, of a bounded lattice $\mathcal{L}$. The partial ordering of $\mathcal{L}$ then serves to compare the truth values of formulas which can be true to some extent. The best-known examples of formal fuzzy logics are probably Monoidal T-norm based Logic (MTL, Esteva and Godo [11]), Basic Logic (BL, Hájek [14]), Gödel logic (G, [13]) and Łukasiewicz logic (Ł, [15]). For all of these logics, which are fully described in terms of axioms, with the modus ponens as deduction rule, soundness and completeness with respect to a corresponding variety[1] can be proved. For instance, a formula can be deduced in MTL iff it is true (i.e., has truth value 1) in every prelinear residuated lattice; recall that a residuated lattice is a structure $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap, \sqcup, *$ and $\Rightarrow$ are binary operators on L and

– $(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,

[1] Recall that a class $\mathcal{K}$ of structures is a variety [14] if there is a set T of identities such that $\mathcal{K}$ is the class of structures in which all identities from T are true.