Uncertainty Modeling by Bilattice-Based Squares and Triangles

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***Abstract*—In this paper, Ginsberg's/Fitting's theory of bilattices, and in particular the associated constructs of bilattice-based squares and triangles, is introduced as an attractive framework for the representation of uncertain and potentially conflicting information, paralleling Goguen's $\mathcal{L}$-fuzzy set theory. We recall some of the advantages of bilattice-based frameworks for handling fuzzy sets and systems, provide the related structures with adequately defined graded versions of the basic logical connectives, and study their properties and relationships.**

***Index Terms*—Bilattices, bilattice-based squares and triangles, implicators, MV-algebras, negators, t-norms and t-conorms.**

I. Introduction

Bilattices are algebraic structures that were introduced by Ginsberg in [1], [2] as a general and uniform framework for a diversity of applications in artificial intelligence. In particular, he treated first-order theories and their consequences, truth maintenance systems, and formalisms for default reasoning. In a series of papers, Fitting then showed that bilattices are very useful tools for providing semantics to logic programs (see, e.g., [3]–[5]), a thesis that was later vindicated in [6]–[8]. Several works have shown that bilattices may serve as a foundation of other areas, such as computational linguistics [9] and distributed knowledge processing [10]. In particular, a family of bilattice-based logics and corresponding proof systems were introduced in [11]–[13], where it was shown that bilattices are useful as the underlying algebraic structures of formalisms for reasoning with imprecise information (see also [14], [15]). This point was recently made explicit in the context of fuzzy set theory, where we have shown (see [16], [17]) that bilattices, and in particular the associated constructs of bilattice-based squares and triangles, provide an elegant framework for bridging between intuitionistic fuzzy sets [18] and interval-valued fuzzy sets [19], [20], two common extensions of fuzzy sets.

The aim of this paper is to substantiate this bilattice-based framework by equiping it with suitable implementations for the common logical connectives of negation, conjunction, disjunction and implication. As is well known from fuzzy set theory, an adequate choice for these operations, inspired by the properties we want them to satisfy, often determines to a great extent the strength of the applications that rely on them.

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Fortunately, we do not have to start our investigation from scratch. Instead, it turns out that ideas from both $\mathcal{L}$-fuzzy set theory [21] and bilattice theory [1], [22] can go a very long way in helping us pinpoint the "best" choice for these connectives, allowing for a positive synergy between the contributing theories. Incidentally, the present paper can also be viewed as a generalization to the lattice-valued and bilattice-valued case of previous papers [23]–[26] that refer to particular forms of 'triangle' and 'square', in which the underlying structure is the unit interval.

The rest of the paper is organized as follows: first, in Section II, we recall some elementary concepts of bilattices and bilattice-based squares and triangles. Section III is the heart of this paper, in which we consider proper representations of logical connectives in our framework: the first part (Section III-A) establishes the representation of involutive negators, the second part (Section III-B) explores the idea of $\mathcal{L}$-representability in the definition of t-norms and t-conorms for modeling conjunction and disjunction, and the last part (Section III-C) introduces several ways of representing implication connectives and examines the relationships among them, as well as their relations to other connectives. In particular, the choice of the "right" negator and the existence of an associated MV-algebra are explored. Finally, in Section IV we hint on the application potential of our bilattice-based framework and conclude.

II. Preliminaries

In this section, we review some basic definitions and notions that pertain to bilattices in general, and bilattice-based squares and triangles in particular. For other expositions of these structures and the motivations behind them, we refer to [16] and [17].

Definition: A *prebilattice* [22] is a structure $\mathcal{B} = (B, \leq_t, \leq_k)$, such that B is a nonempty set containing at least two elements, and $(B, \leq_t)$, $(B, \leq_k)$ are complete lattices. A *bilattice* [1] is a structure $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$, such that $(B, \leq_t, \leq_k)$ is a prebilattice, and $\neg$ (the "negation") is a unary operation on B satisfying, for every x, y in B, the following properties:

1) $\neg\neg x = x$;
2) if $x \leq_t y$ then $\neg x \geq_t \neg y$;
3) if $x \leq_k y$ then $\neg x \leq_k \neg y$.

In the sequel, following the usual notations for the basic bilattice operations, we shall denote by $\wedge$ (respectively, by $\vee$) the $\leq_t$-meet (the $\leq_t$-join) and by $\otimes$ (respectively, by $\oplus$) the $\leq_k$-meet (the $\leq_k$-join) of a bilattice $\mathcal{B}$. f and t denote the $\leq_t$-extreme elements, and $\perp$, $\top$ denote the $\leq_k$-extreme elements of $\mathcal{B}$. Intuitively, these elements can be perceived as "false," "true," "unknown" (i.e., neither true nor false) and

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