Intuitionistic Fuzzy Connectives Revisited

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Abstract

In this paper we examine in detail the definition of intuitionistic fuzzy (IF) connectives: negation, conjunction, disjunction and implication; we argue that some of the existing definitions that appear in the IF literature are not sufficiently general for all practical purposes, and suggest to replace them with new ones. One direct example to this effect will be the construction of an IF implicator that satisfies all Smets–Magrez axioms and that in the classical framework could not be expressed. Our approach will be algebraic: we treat IF connectives as lattice–valued mappings, and we translate desirable logical properties into algebraic equations to be satisfied.

Keywords: intuitionistic fuzzy t–norms, t–conorms, negators, implicators, Smets–Magrez axioms.

1 Introduction

The objective of this paper is to equip intuitionistic fuzzy set (IFS) theory, and in particular the definition of its connectives, with robust algebraic foundations. IF connectives are used to combine existing propositions in IF logics [1] into compound propositions, and equivalently they serve to implement operations like complement, intersection and union on IFS's.

The defining idea behind our approach is to treat logical connectives as algebraic mappings; since we want them to act on IF truth values, that is: ordered pairs (μ, ν) satisfying $\mu+\nu \leq 1$ (μ is often called the degree of truth, while ν conversely denotes the degree of non–truth), to describe the domain and codomain structure the partially ordered set $(L^*, \leq_{L^*})$ was introduced in [3]:

Definition 1.1 (Partially ordered set $(L^*, \leq_{L^*})$)

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2$$

It is easily verified that $(L^*, \leq_{L^*})$ is a complete lattice. By $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ we denote its bounds.

2 Negation

Atanassov [1] defined the negation of an element $(x_1, x_2) \in L^*$ as (x_2, x_1). In [2] a more general definition encapsulating the former was given:

Definition 2.1 (IF Negator) *An IF negator is any decreasing $L^* \to L^*$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}, \mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x, \forall x \in L^*$, $\mathcal{N}$ is called an involutive IF negator.*

The mapping $\mathcal{N}_s$, defined as $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$ will be called the *standard negator.*

Bustince, Kacprzyk and Mohedano have already in [2] made a tremendous contribution to our knowledge about IF negators. In particular, they proved that given a decreasing $[0,1] \to [0,1]$ mapping ϕ satisfying $\phi(0)=1$ and $\phi(a) \leq 1-a$ for all $a \in [0,1]$, $\mathcal{N}_\phi$ defined by $\mathcal{N}_\phi(x_1,x_2) = (\phi(1-x_2), 1-\phi(x_1))$ is an IF negator. Also, they proved that $\mathcal{N}_\phi$ is involutive as soon as ϕ is involutive.

We have studied the other direction, namely: given an involutive IF negator $\mathcal{N}$, does there exist a $[0,1] \to [0,1]$ mapping ϕ such that $\mathcal{N}(x_1,x_2)$ can be written as $(\phi(1-x_2), 1-\phi(x_1))$ for all $(x_1,x_2) \in L^*$? To this effect, we proved the following lemmata and theorem:

Lemma 2.1 *For any involutive IF negator $\mathcal{N}$ on L^* it holds that $\mathcal{N}(0,0)=(0,0)$.*

Corollary 2.1 *For any involutive IF negator $\mathcal{N}$ on L^* there holds, for all $a \in [0,1]$: $pr_2\mathcal{N}(0,a) = 0$ and $pr_1\mathcal{N}(a,0)=0$, where pr_1 and pr_2 denote the first and second projection mapping on L^*, defined as $pr_1(x_1,x_2)=x_1$ and $pr_2(x_1,x_2)=x_2$.*

Lemma 2.2 *Let $\mathcal{N}$ be an involutive IF negator on L^*. For all $x=(x_1,1-x_1)$ and $x'=(x_1,0) \in L^*$, $pr_2\mathcal{N}(x) = pr_2\mathcal{N}(x')$. Similarly, if $x=(1-x_2,x_2)$ and $x'=(0,x_2) \in L^*$, then $pr_1\mathcal{N}(x) = pr_1\mathcal{N}(x')$.*

Lemma 2.3 *Let $\mathcal{N}$ be an involutive IF negator on L^*. For $x=(x_1,1-x_1) \in L^*$ it holds that $pr_1\mathcal{N}(x) + pr_2\mathcal{N}(x) = 1$.*

Theorem 2.1 *Let $\mathcal{N}$ be an involutive IF negator on L^*, and let the $[0,1] \to [0,1]$ mapping n be defined by, for $a \in [0,1]$, $n(a) = pr_1\mathcal{N}(a,1-a)$. Then for all $(x_1,x_2) \in L^*$: $\mathcal{N}(x_1,x_2) = (n(1-x_2), 1-n(x_1))$.*

It can be verified that n is decreasing, involutive and satisfies $n(0)=1$. In other words, every involutive IF negator induces an involutive fuzzy negator.[1]

[1] But not vice versa: Bustince et al. [2] showed that e.g. the fuzzy negators (due to Sugeno) defined by $N_\lambda(a) = \frac{1-a}{1+\lambda a}$ where $-1 < \lambda < 0$ are involutive but $(\exists(x_1,x_2) \in L^*)((N_\lambda(1-x_2), 1-N_\lambda(x_1)) \notin L^*)$.

3 Conjunction and Disjunction

Atanassov [1] defined the conjunction (resp. disjunction) of two elements (x_1,x_2) and $(y_1,y_2) \in L^*$ as $(\min(x_1,x_2),\max(x_1,x_2))$ (resp. $(\max(x_1,x_2),\min(x_1,x_2))$). In [3] we noted that this definition can be generalized using a t–norm[2] T and a t–conorm[3] S, provided $T(a,b) \leq 1-S(1-a,1-b)$ for all $(a,b) \in L^*$. The generalization should not stop at this stage, however, as will be demonstrated in this section.

Since $(L^*, \leq_{L^*})$ is a partially ordered set, an order–theoretic definition of IF conjunction and disjunction arises naturally. To this aim, we introduce IF t–norms and IF t–conorms.

Definition 3.1 (IF Triangular Norm) *An IF t–norm is any increasing, commutative, associative $L^* \to L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*},x)=x$, for all $x \in L^*$.*

Definition 3.2 (IF Triangular Conorm) *An IF t–conorm is any increasing, commutative, associative $L^* \to L^*$ mapping $\mathcal{S}$ satisfying $\mathcal{S}(0_{L^*},x)=x$, for all $x \in L^*$.*

IF t–norms and t–conorms can be easily generated using their classical counterparts, as the following claim proves: [3]

Theorem 3.1 *Given a fuzzy t–norm T and t–conorm S satisfying $T(a,b) \leq 1-S(1-a,1-b)$ for all $a,b \in [0,1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by, for $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in L^*:*

$$\begin{aligned}\mathcal{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2)),\\ \mathcal{S}(x,y) &= (S(x_1,y_1), T(x_2,y_2)),\end{aligned}$$

are an IF t–norm and an IF t–conorm, respectively.

On the other hand, we do not have that for every IF t–norm (IF t–conorm) there exist

[2] A t–norm is any increasing, commutative, associative $[0,1] \times [0,1] \to [0,1]$ mapping T satisfying $T(1,a)=a$ for every $a \in [0,1]$

[3] A t–conorm is any increasing, commutative, associative $[0,1] \times [0,1] \to [0,1]$ mapping S satisfying $S(0,a)=a$ for every $a \in [0,1]$

a t–norm and t–conorm such that the above equalities hold. We therefore need to introduce an additional definition:

Definition 3.3 (t–representability) *An IF t–norm $\mathcal{T}$ is called t–representable if there exist a t–norm T and t–conorm S such that $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$ for all $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in L^*. An IF t–conorm $\mathcal{S}$ is called t–representable if there exist a t–norm T and t–conorm S such that $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$ for all $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in L^*. In both cases, we say that T and S are representants of the IF t–(co)norm.*

We proceed to introduce an IF t–conorm that is not t–representable:

Theorem 3.2 *The $(L^*)^2 \to L^*$ mapping $\mathcal{S}_1$ defined as, for $x,y \in L^*$,*

$$\mathcal{S}_1(x,y) = \begin{cases} x & \text{if } y = 0_{L^*} \\ y & \text{if } x = 0_{L^*} \\ (\max(1-x_2, 1-y_2), & \\ \min(x_2,y_2)) & \text{else} \end{cases}$$

is a non t–representable IF t-conorm.

It can easily be verified that $\mathcal{S}_1$ is an IF t–conorm. For $\mathcal{S}_1$ to be t–representable, there must exist a fuzzy t–conorm S and a fuzzy t–norm T such that, for all $x = (x_1,x_2), y = (y_1,y_2) \in L^*$, $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$. Suppose that such S and T exist, and let $x = (0.1, 0.6)$, $x' = (0.1, 0.7)$ and $y = (0.1, 0.8)$. From $\mathcal{S}_1(x,y) = (0.4, 0.6)$ we obtain $S(x_1,y_1) = 0.4$. On the other hand, from $\mathcal{S}_1(x',y) = (0.3, 0.7)$ it follows that $S(x'_1,y_1) = S(x_1,y_1) = 0.3$, which is obviously in contradiction with the above information, so we must conclude that S does not exist, and hence $\mathcal{S}_1$ is not t–representable.

Going one step further, the following theorem introduces an IF t–conorm that is continuous but even not t–representable.

Theorem 3.3 *The $(L^*)^2 \to L^*$ mapping $\mathcal{S}_2$ defined as, for $x,y \in L^*$,*

$$\mathcal{S}_2(x,y) = (\min(1, x_1+1-y_2, y_1+1-x_2), \max(0, x_2+y_2-1))$$

is a continuous, non t–representable IF t–conorm.

The dual of an IF t–norm $\mathcal{T}$ (IF t–conorm $\mathcal{S}$) w.r.t. an IF negator $\mathcal{N}$ is the mapping $\mathcal{T}^*$ (resp. $\mathcal{S}^*$) defined by, for $x,y \in L^*$, $\mathcal{T}^*(x,y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x),\mathcal{N}(y)))$ (resp. $\mathcal{S}^*(x,y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x),\mathcal{N}(y)))$). It can be verified that $\mathcal{T}^*$ is an IF t–conorm and $\mathcal{S}^*$ is an IF t–norm.

The dual IF t–norm of $\mathcal{S}_1$ w.r.t. the standard negator $\mathcal{N}_s$ is $\mathcal{T}_1$ defined as, for $x,y \in L^*$, $\mathcal{T}_1(x,y) = \mathcal{N}_s(\mathcal{S}_1(\mathcal{N}_s(x),\mathcal{N}_s(y)))$, or:

$$\mathcal{T}_1(x,y) = \begin{cases} x & \text{if } y = 1_{L^*} \\ y & \text{if } x = 1_{L^*} \\ (\min(x_1,y_1), & \\ \max(1-x_1, 1-y_1)) & \text{else} \end{cases}$$

The dual IF t–norm of $\mathcal{S}_2$ w.r.t. $\mathcal{N}_s$ is $\mathcal{T}_2$ defined as, for $x,y \in L^*$,

$$\mathcal{T}_2(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+1-y_1, y_2+1-x_1))$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are also not t–representable, as the following theorem implies:

Theorem 3.4 *The dual IF t–norm with respect to an involutive IF negator $\mathcal{N}$ on L^* of a t–representable IF t–conorm is t–representable. The dual IF t–conorm with respect to an involutive IF negator $\mathcal{N}$ on L^* of a t–representable IF t–norm is t–representable.*

4 Implication

A very general definition of the implication connective (denoted IF implicator) is given in the following definition [3]:

Definition 4.1 (IF Implicator) *An IF implicator is any $(L^*)^2 \to L^*$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0_{L^*},0_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*},0_{L^*}) = 0_{L^*}, \mathcal{I}(0_{L^*},1_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*},1_{L^*}) = 1_{L^*}$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component.*

Two important subclasses of IF implicators were introduced in [5]. It is easily verified that each of the mappings defined hereafter is indeed an IF implicator in the sense of definition 4.1.

Definition 4.2 (IF S–implicator) *Let $\mathcal{S}$ be an IF t–conorm and $\mathcal{N}$ an IF negator. The IF S–implicator generated by $\mathcal{S}$ and $\mathcal{N}$ is the mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y)$$

If $\mathcal{S}$ is t–representable, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is called a t–representable IF S–implicator.

Definition 4.3 (IF R–implicator) *Let $\mathcal{T}$ be an IF t–norm. The IF R–implicator generated by $\mathcal{T}$ is the mapping $\mathcal{I}_{\mathcal{T}}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\gamma \in L^* \mid \mathcal{T}(x,\gamma) \leq_{L^*} y\}$$

If $\mathcal{T}$ is t–representable, $\mathcal{I}_{\mathcal{T}}$ is called a t–representable IF R–implicator.

The suitability of IF implicators for a variety of purposes can be assessed using the (generalized) criteria introduced by Smets and Magrez in [6]:

Definition 4.4 (Axioms of Smets and Magrez for an IF implicator $\mathcal{I}$)

- *(A.1)* $(\forall y \in L^*)(\mathcal{I}(.,y)$ *is decreasing)* $(\forall x \in L^*)(\mathcal{I}(x,.)$ *is increasing)* (monotonicity laws)
- *(A.2)* $(\forall x \in L^*)(\mathcal{I}(1_{L^*}, x) = x)$ (neutrality principle)
- *(A.3)* $(\forall (x,y) \in (L^*)^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)))$ (contrapositivity w.r.t. an IF negator $\mathcal{N}$)
- *(A.4)* $(\forall (x,y,z) \in (L^*)^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))$ (interchangeability principle)
- *(A.5)* $(\forall (x,y) \in (L^*)^2)(x \leq_{L^*} y \iff \mathcal{I}(x,y) = 1_{L^*})$ (confinement principle)
- *(A.6)* $\mathcal{I}$ *is continuous* (continuity)

In [5] we discovered that there does not exist a t–representable IF S–implicator nor a t–representable IF R–implicator that satisfies all Smets–Magrez axioms at once. We also proved that an IF implicator satisfying (A.1), (A.2), (A.3) and (A.4)—which is also called an IF model implicator—must necessarily be an IF S–implicator. It is a great surprise, therefore, that the conjecture made in [5] inspired by the above facts, namely that there does not exist an IF implicator satisfying all Smets–Magrez axioms, is convincingly falsified by the following theorem:

Theorem 4.1 *The IF S–implicator $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}}$ generated by $\mathcal{S}_2$ and $\mathcal{N}$ satisfies (A.1), (A.2), (A.3), (A.4) and (A.6). $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}}$ satisfies (A.5) if and only if $\mathcal{N} = \mathcal{N}_s$. In that case, $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}_s} = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$.*

Moreover, $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}}$ is also an IF R–implicator, and when applied to fuzzy values $(x, 1 - x) \in L^*$ it coincides with the well–known Łukasiewicz fuzzy implicator, defined by, for $a, b \in [0,1]$, $I_L(a,b) = \min(1, 1 - a + b)$.

Theorem 4.2 *The IF R–implicator $\mathcal{I}_{\mathcal{T}_2}$ generated by the IF t–conorm $\mathcal{T}_2$ equals the IF S–implicator $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}_s}$.*

Theorem 4.3 *The IF S–implicator $\mathcal{I}_{\mathcal{S}_2,\mathcal{N}_s}$ is an extension of the Łukasiewicz fuzzy implicator I_L.*

Note that in fuzzy set theory I_L is an S– and R–implicator[4] satisfying (the fuzzy version of) the Smets–Magrez axioms. It is all the more compelling to note, however, that a naive extension of I_L to an IF R–implicator, obtained by

- the observation that I_L is a fuzzy R–implicator generated by T_W, the Łukasiewicz t–norm (defined by $T_W(a,b) = \max(0, a + b - 1)$);

[4]The fuzzy S–implicator generated by a t–conorm S is the mapping I_S defined as, for all $x, y \in [0,1]$:

$$I_S : \begin{array}{ccc} [0,1]^2 & \to & [0,1] \\ (x,y) & \mapsto & S(1-x, y) \end{array}$$

The fuzzy R–implicator generated by a t–norm T is the mapping I_T defined as, for all $x, y \in [0,1]$:

$$I_T : \begin{array}{ccc} [0,1]^2 & \to & [0,1] \\ (x,y) & \mapsto & \sup\{\gamma \in [0,1] | T(x,\gamma) \leq y\} \end{array}$$

- the application of theorem 3.1 to T_W and its dual t–conorm the bounded sum S_{+_b} (defined by $S_{+_b}(a,b) = \min(1, a+b)$), yielding the IF t–norm $\mathcal{T}$ defined by $\mathcal{T}((x_1,x_2),(y_1,y_2)) = (T_W(x_1,y_1), S_{+_b}(x_2,y_2)))$;

- using $\mathcal{T}$ to generate $\mathcal{I}_{\mathcal{T}}((x_1,x_2),(y_1,y_2)) = (\min(1, 1+y_1-x_1, 1+x_2-y_2), \max(0, y_2-x_2))$;

is t–representable, differs from $\mathcal{I}_{\mathcal{T}_2}$ only by substituting x_2 with $1-x_1$ in the second component of $\mathcal{I}_{\mathcal{T}}$ but does not satisfy (A.3).

5 Conclusion

In this paper, we re–investigated the definition of the most important connectives in IFS theory: those for negation, conjunction, disjunction and implication. A representation theorem for involutive IF negators complementary to Bustince et al. 's work was obtained, while an order–theoretic approach, strictly more general than the existing one, to the definition of conjunction and disjunction turned out to be indispensable when looking for a suitable candidate for the implication connective–i.e. an IF implicator satisfying the combined Smets–Magrez axioms.

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References

[1] K. T. ATANASSOV (1999). Intuitionistic Fuzzy Sets. Physica–Verlag, Heidelberg – New York, 1999.

[2] H. BUSTINCE, J. KACPRZYK, V. MOHEDANO (2000). Intuitionistic Fuzzy Generators—Application to Intuitionistic Fuzzy Complementation. In *Fuzzy Sets and Systems*, volume 114, pages 485–504, 2000.

[3] C. CORNELIS, G. DESCHRIJVER (2001). The Compositional Rule of Inference in an Intuitionistic Fuzzy Logic Framework. In *Proceedings of ESSLLI 2001 Student Session, (Kristina Striegnitz, ed.)*, Kluwer Academic Publishers, pages 83–94, August 2001.

[4] C. CORNELIS, E. E. KERRE (2001). Reliability of Information: Motivation for an Intuitonistic Theory of Possibility. Submitted to Fuzzy Sets and Systems, October 2001.

[5] C. CORNELIS, G. DESCHRIJVER, E. E. KERRE (2002). Classification of Intuitionistic Fuzzy Implicators: an Algebraic Approach. Accepted to *Special Session on Intuitionistic Fuzzy Sets: Theory and Applications*, Durham, United States, March 2002.

[6] P. SMETS, P. MAGREZ (1987). Implication in fuzzy logic. In *International Journal of Approximate Reasoning*, volume 1, pages 327–347, 1987.