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Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application

Chris Cornelis *, Glad Deschrijver, Etienne E. Kerre

Fuzziness and Uncertainty Modelling Research Unit, Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281 (S9), 9000 Gent, Belgium

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Abstract

With the demand for knowledge-handling systems capable of dealing with and distinguishing between various facets of imprecision ever increasing, a clear and formal characterization of the mathematical models implementing such services is quintessential. In this paper, this task is undertaken simultaneously for the definition of implication within two settings: first, within intuitionistic fuzzy set theory and secondly, within interval-valued fuzzy set theory. By tracing these models back to the underlying lattice that they are defined on, on one hand we keep up with an important tradition of using algebraic structures for developing logical calculi (e.g. residuated lattices and MV algebras), and on the other hand we are able to expose in a clear manner the two models' formal equivalence. This equivalence, all too often neglected in literature, we exploit to construct operators extending the notions of classical and fuzzy implication on these structures; to initiate a meaningful classification framework for the resulting operators, based on logical and extra-logical criteria imposed on them; and finally, to re(de)fine the intuititive ideas giving rise to both approaches as models of imprecision and apply them in a practical context.

* Corresponding author. Tel.: +32-9-264-47-72; fax: +32-9-264-49-95.

E-mail addresses: chris.cornelis@ugent.be (C. Cornelis), glad.deschrijver@ugent.be (G. Deschrijver), etienne.kerre@ugent.be (E.E. Kerre).

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1. Introduction

Intuitionistic fuzzy sets [1] and interval-valued fuzzy sets ([54,67] and more recently, [58]) are two intuitively straightforward extensions of Zadeh's fuzzy sets [66], that were conceived independently to alleviate some of the drawbacks of the latter. Henceforth, for notational ease, we abbreviate "intuitionistic fuzzy set" to IFS and "interval-valued fuzzy set" to IVFS. IFS theory basically defies the claim that from the fact that an element x "belongs" to a given degree (say μ) to a fuzzy set A, naturally follows that x should "not belong" to A to the extent $1 - \mu$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element of the universe both a degree of membership μ and one of non-membership ν such that $\mu + \nu \leqslant 1$, thus relaxing the enforced duality $\nu = 1 - \mu$ from fuzzy set theory. Obviously, when $\mu + \nu = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered. IFSs owe their name [4] to the fact that this latter identity is weakened into an inequality, in other words: a denial of the law of the excluded middle occurs, one of the main ideas of intuitionism. [1]

IVFS theory emerged from the observation that in a lot of cases, no objective procedure is available to select the crisp membership degrees of elements in a fuzzy set. It was suggested to alleviate that problem by allowing to specify only an interval $[\mu_1, \mu_2]$ to which the actual membership degree is assumed to belong. A related approach, second-order fuzzy set theory, also introduced by Zadeh [67], goes one step further by allowing the membership degrees themselves to be fuzzy sets in the unit interval; this extension is not considered in this paper.

Both approaches, IFS and IVFS theory, have the virtue of complementing fuzzy sets, that are able to model *vagueness*, with an ability to model *uncertainty* as well. [2] IVFSs reflect this uncertainty by the length of the interval membership degree $[\mu_1, \mu_2]$, while in IFS theory for every membership degree

[1] The term "intuitionistic" is to be read in a "broad" sense here, alluding loosely to the denial of the law of the excluded middle on element level (since $\mu + \nu < 1$ is possible). A "narrow", graded extension of intuitionistic logic proper has also been proposed and is due to Takeuti and Titani [57]—it bears no relationship to Atanassov's notion of IFS theory.

[2] In these pages, we juxtapose "vagueness" and "uncertainty" as two important aspects of imprecision. Some authors [45,47,60] prefer to speak of "non-specificity" and reserve the term "uncertainty" for the global notion of imprecision.